(*b*) We can place the two scales at right angles and show the correspondence, as in Figs. 1 or 2.

The teacher is then ready to explain Figs. 3 and 4. The student should be led to see the advantages of Fig. 4: from it we can approximate new corresponding values that would be impossible from a table of data or from Fig. 2 or 3. A figure like 1 may look ludicrous on the blackboard, and certainly no textbook has yet dared to contain one, but no one will deny that it illustrates a significant step in the evolution of the graph. The teacher must make sure that the student understands that the two lines (axes, as we later call them)

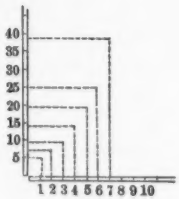


FIG. 3.

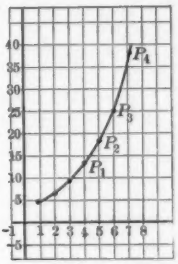


FIG. 4.

need not intersect at the zero point, that with a change of units the graph will look different, that if the problem hinges on the correspondence only over a particular range then the zeros may not even appear on either of the axes. All such matters as tend to emphasize *correspondence* should be considered at this point, and considered fully. It may even be useful to illustrate discontinuous correspondences, multiple valued ones, etc. Further, either here or before beginning II, the student should have some drill in finding functional relations, *i. e.*, stating by an equation a relation given descriptively. Problems involving motion or the measurement of physical magnitudes are the simplest and most useful.

If presenting the notion of function were all that we wish to do we could satisfactorily stop here. And we note that we are able to explain this concept without referring specifically to axes or coördinates. In truth, I believe the value of this mode of presentation lies in the very fact that coördinates and graphing have been kept in the background while the student is studying relations between variables. This is the reverse of what the textbooks do, for they consider coördinates first and then later, preliminary to locus problems, study functional relations. This reversal of order has probably arisen through the teaching of graphing in the high schools. Graphing is a very easy idea for the high school student to grasp; it enlightens the subject of simultaneous linear equations, and the teacher is glad to introduce a geometric idea into algebraic work. As a correlation of geometry and algebra such work is useful, but as an introduction to the notion of function it is misleading.

We can, however, now introduce coördinates in such a fashion that they will appear not at all as a *new* idea, but will appear as *that element which is common to all* the previous exercises. Pedagogically the introduction of coördinates at this stage is proper in that we now focus the student's attention on the underlying processes of his work. And since the student has been unconsciously using them, nothing further is needed except to define the technical terms by which the ideas will be referred to in the future.

III. We have used letters to denote one value from a set. We have axes, units, and points in the plane whose projections are the values which correspond. Conversely, every pair of corresponding values fixes a point in the plane. The numbers which determine the point we now choose to call coördinates. And thus we proceed to develop the rest of the terminology used in the study of relations $f(x, y) = 0$.

The work thus far can be done completely in three days, or even in two, depending upon how much the teacher wishes to drill on the various problems. All that is essential in explaining the notions of function, coördinates, graphing has been covered. If, however, an additional day is available, then some exercises involving the construction of such curves as $xy = 1$, $y = x^3$ are very profitable. While many papers have been published on the use of cross section paper for curve tracing I do not know that any one has pointed out its pedagogic value.

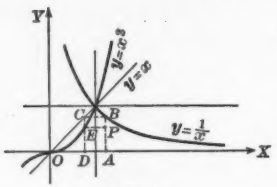


FIG. 5.

As a single illustration let us trace the thoughts of the student when asked for a method for constructing $x^3y = 1$ (see Fig. 5). He writes $y = 1/x^3$ and sees that he is dealing with the reciprocal of a variable and with the cube of a variable. Consequently he must first draw $y = 1/x$ and $y = x^3$. He selects any point A, writes $OA = x$, $AB = 1/x = DC = OD$, and finally $DE = \overline{OD}^3 = 1/x^3$. Consequently P is a point on the desired curve. The reader will see that this problem contains every idea associated with the notion of a function: the length of AB varies with the choice of A but varies according to a definite law; to cube AB it must be laid off horizontally from the origin; P and not E is the desired point on the curve because the ordinate is measured vertically from the end A of the abscissa; and finally, each equation of the problem states like a law the length of one line as compared with another. As the attention shifts from A to B to C, D, E, P the student must trace the fortunes of the variable as it is influenced by the different laws.

Whether the student thinks exactly in this fashion or not, this type of exercise is more valuable than the kind wherein he mechanically computes data for points and draws the curve through them. Even when the student can not himself discover the construction, nevertheless the subsequent reproduction of the teacher's method will be valuable if the student is required to give the reasons for each step. Also, this type of exercise more than any other affords good opportunity for questions by the teacher.

DIRECTED ANGLES AND INVERSION, WITH A PROOF OF SCHOUTE'S THEOREM.

By ROGER A. JOHNSON, Western Reserve University.

In this paper it is proposed to apply the method of directed angles[1] to the theory of inversion, especially to the use of the latter in studying properties of the triangle. None of the theorems below is new, but in most cases the form of presentation is new, and this form seems to be more definite and more easily workable than the usual statements.

We begin with a group of fundamental angle relations connected with the transformation of inversion; these are then applied to the triangle, yielding various interesting theorems. Thirdly, we state some of the best known properties of the so-called circles of Apollonius, and the circles of Schoute which are orthogonal to them, and finally we have a simple proof of a noted theorem of Schoute. This proof resembles to some extent a proof given by W. Gallatly,[2] but avoids some complexities. The original proof given by Schoute[3] was algebraic.

We recall the most useful properties of directed angles from the earlier paper:[4]

(*A*) The addition of directed angles follows the same rules as that of directed segments of a line. A useful identity is

$$\measuredangle l_1, l_2 + \measuredangle l_3, l_4 = \measuredangle l_1, l_4 + \measuredangle l_3, l_2.$$

(*B*) Three points A, B, C are collinear if and only if

$$\measuredangle ACB = 0$$

or, in other words, if for any other point D

$$\measuredangle ACD = \measuredangle BCD.$$

(*C*) Four points A, B, C, D lie on a circle or line if and only if

$$\measuredangle ABC = \measuredangle ADC.$$

(*D*) The Miquel formula for the angles of the pedal triangle of a point P with regard to triangle $A_1A_2A_3$. If PP_1, PP_2, PP_3 are the $\perp$s to the sides, then

$$\measuredangle A_2PA_3 = \measuredangle P_2P_1P_3 + \measuredangle A_2A_1A_3, \text{ etc.}$$

---

[1] See a paper by R. A. Johnson, *Directed Angles in Elementary Geometry*, this MONTHLY, Vol. XXIV, page 101, 1917.

[2] *The Modern Geometry of the Triangle*, Chap. VIII.

[3] Schoute, *Over een neawer verband tusschen hoek en cirkel van Brocard*, Amsterdam Acad. Proceedings, Series III, vol. 3, 1887, page 22; see Coolidge, *Geometry of the Circle and the Sphere*, page 127.

[4] One who desires to extract the essence of the present note without using the method of directed angles may proceed by drawing in each case a figure, and interpreting any statement of the form $\measuredangle x = \measuredangle y$ as meaning that the angles x and y of the figure are either equal or supplementary or differ by 180°.

Consider now an inversion with regard to a fixed circle with center O, radius r.

THEOREM 1. *Two pairs of inverse points are either collinear with O, or concyclic on a circle orthogonal to the circle of inversion.*

For

$$\overline{OP} \cdot \overline{OP'} = \overline{OQ} \cdot \overline{OQ'} = r^2.$$

THEOREM 2. *Triangles OPQ, $OQ'P'$ are inversely similar.*

For angle O is common; and by (C) and theorem 1 we have

$$\measuredangle OPQ = \measuredangle P'Q'O, \qquad \measuredangle PQO = \measuredangle OP'Q'.$$

COROLLARY. *The distance between two points, and the distance between their inverses, are connected by*

$$\overline{P'Q'} = \overline{QP} \frac{r^2}{\overline{OP} \cdot \overline{OQ}}.$$

THEOREM 3. *If P, Q, R are any three points, P', Q', R' their inverses,*

$$\measuredangle QPR + \measuredangle Q'P'R' = \measuredangle QOR.$$

Proof.

$$\measuredangle QPO = \measuredangle OQ'P',$$

$$\measuredangle OPR = \measuredangle P'R'O.$$

Adding,

$$\measuredangle QP, PO + \measuredangle PO, PR = \measuredangle OQ', Q'P' + \measuredangle Q'R', R'O,$$

or, applying (A),

$$\measuredangle QPR = \measuredangle R'P'Q' + \measuredangle Q'OR',$$

$$\measuredangle QPR + \measuredangle Q'P'R' = \measuredangle QOR.$$

This formula is exceedingly powerful; in fact, it may be ranked in point of importance with the Miquel formula (D), which it resembles in form. From it many of the important properties of inversion are immediate corollaries.

COROLLARY 1. *For any four points and their inverses,*

$$\measuredangle PQR + \measuredangle RSP = \measuredangle R'Q'P' + \measuredangle P'S'R'.$$

COROLLARY 2. *The inverse of a circle not passing through the center of inversion is also a circle.*

COROLLARY 3. *The inverse of a straight line not through the center of inversion is a circle through that point, and conversely.*

COROLLARY 4. *The angle of intersection of two circles is the same as that of the inverse circles. This is true also for any intersecting curves.*

We turn now to the triangle.

THEOREM 4. *If two points are mutually inverse with regard to the circumcircle of a triangle $A_1A_2A_3$, their pedal triangles are inversely similar.*

For each vertex of the triangle is self-inverse, hence we have from Theorem 3,

$$\measuredangle A_2PA_3 + \measuredangle A_2P'A_3 = \measuredangle A_2OA_3.$$

Applying (D),

$$\measuredangle A_2A_1A_3 + \measuredangle P_2P_1P_3 + \measuredangle A_2A_1A_3 + \measuredangle P_2'P_1'P_3' = \measuredangle A_2OA_3.$$

But

$$2\measuredangle A_2A_1A_3 = \measuredangle A_2OA_3;$$

hence

$$\measuredangle P_2P_1P_3 = \measuredangle P_3'P_1'P_2'.$$

Since all the angles of the pedal triangles satisfy such equations, the triangles are inversely similar.

We may regard as a corollary to this theorem the fact (otherwise much more easily proved) that the pedal triangle of a point on the circumcircle reduces to a line.

THEOREM 5. *If an inversion is performed with any center P, the inverses B_1, B_2, B_3 of any three points A_1, A_2, A_3 form a triangle similar to the pedal triangle of P with regard to $\triangle A_1A_2A_3$.*

For since

$$\measuredangle A_2PA_3 = \measuredangle A_2A_1A_3 + \measuredangle B_2B_1B_3 \qquad \text{(Theorem 3)}$$

$$\measuredangle A_2PA_3 = \measuredangle A_2A_1A_3 + \measuredangle P_2P_1P_3 \qquad (D)$$

it follows that

$$\measuredangle P_2P_1P_3 = \measuredangle B_2B_1B_3, \quad \text{etc.}$$

COROLLARY. *If a triangle is subjected to an inversion with regard to two points which are inverse with regard to the circumcircle, the resulting triangles are inversely similar.*

THEOREM 6. *There is a single point which may be taken as center of inversion, in order that a given triangle may be transformed into a triangle similar in a given sense to a given triangle.*

For the center of inversion P is located uniquely by the equations

$$\measuredangle A_2PA_3 = \measuredangle A_2A_1A_3 + \measuredangle B_2B_1B_3,$$

$$\measuredangle A_3PA_1 = \measuredangle A_3A_2A_1 + \measuredangle B_3B_2B_1.$$

An exceptional case arises when the given triangles are inversely similar.

THEOREM 7. *If four points are subjected to an inversion, the pedal triangle of one of the points with regard to the other three is inversely similar to the corresponding pedal triangle for the inverted points.*

Let an inversion carry A_1, A_2, A_3, P into B_1, B_2, B_3, Q respectively; and let the pedal triangle of P with regard to $A_1A_2A_3$ be $P_1P_2P_3$, and that of Q with regard to $B_1B_2B_3$ be $Q_1Q_2Q_3$. Then

$$\measuredangle A_2PA_3 + \measuredangle B_2QB_3 = \measuredangle A_2OA_3,$$

$$\measuredangle A_2A_1A_3 + \measuredangle P_2P_1P_3 + \measuredangle B_2B_1B_3 + \measuredangle Q_2Q_1Q_3 = \measuredangle A_2OA_3.$$

But

$$\measuredangle A_2A_1A_3 + \measuredangle B_2B_1B_3 = \measuredangle A_2OA_3$$

so that

$$\measuredangle P_2P_1P_3 + \measuredangle Q_2Q_1Q_3 = 0,$$

as was to be proved.

We pass now to the circles of Apollonius and the isodynamic points. The locus of a point whose distances from two vertices of a triangle are proportional to the adjacent sides, is a circle through the third vertex. In any triangle there are three such circles, called the circles of Apollonius. They are well known, and we shall merely mention a few of their properties. It is easily proved that the center of the circle through A_1 lies on A_2A_3; that the circle passes through the feet of the bisectors of angle A_1; and that the tangent to the circumcircle at A_1 passes through its center. The three circles intersect at two points, called isodynamic points; the distances of these from the sides of the triangle are inversely as the lengths of these sides.

THEOREM 8. *If two points are mutually inverse with regard to the circle of*

Apollonius C_1 through A_1, their pedal triangles are inversely similar in the sense

$$\triangle P_1P_2P_3 \simeq \triangle Q_1Q_2Q_2.$$

First, A_2 and A_3 are inverse with regard to this circle, for if L_1 is its center, $\overline{L_1A_2}\cdot\overline{L_1A_3} = \overline{L_1A_1}^2$, since L_1A_1 is tangent to the circumcircle. Hence we have two pairs of inverse points, A_2, A_3 and P, Q, and further A_1 is inverse to itself

$$\measuredangle A_1PA_2 + \measuredangle A_1QA_3 = \measuredangle A_1L_1A_3.$$

Applying (*D*),

$$\measuredangle A_1A_3A_2 + \measuredangle P_1P_3P_2 + \measuredangle A_1A_2A_3 + \measuredangle Q_1Q_3Q_2 = \measuredangle A_1A_3A_2 + \measuredangle A_1A_2A_3,$$

whence

$$\measuredangle P_1P_3P_2 = \measuredangle Q_3Q_2Q_1.$$

Similarly,

$$\measuredangle P_3P_2P_1 = \measuredangle Q_1Q_3Q_2.$$

COROLLARY. *If P lies on the circle C_1, its pedal triangle is isosceles in the sense $\overline{P_1P_2} = \overline{P_1P_3}$. The pedal triangle of each isodynamic point is equilateral. It follows that the isodynamic points are inverse with regard to the circumcircle.*

This last result may also be deduced from the fact that these points are the intersections of circles orthogonal to the circumcircle.

Also orthogonal to the circles of Apollonius is their line of centers. This may easily be shown to be the polar, with regard to the circumcircle, of the symmedian point K; therefore the inverse of this line, which is the Brocard circle, is also orthogonal to the circles of Apollonius. We may designate the coaxial system which includes the circumcircle, the Brocard circle, the Lemoine line (polar of the symmedian point) and the isodynamic points, as the Schoute system of circles. Its members are orthogonal to the circles of Apollonius.

THEOREM 9. *If an inversion is performed with regard to a circle about either isodynamic point, the given triangle transforms into an equilateral triangle whose center is at the inverse of the other isodynamic point, the circles of Apollonius invert into the altitudes of this triangle, and the circles of Schoute into circles concentric with it.*

THEOREM 10. *In an equilateral triangle, the locus of a point whose pedal triangle has a constant Brocard angle is a circle concentric with the triangle.*

For let $B_1B_2B_3$ be an equilateral triangle whose side is b; let O denote its center; let $P_1P_2P_3$ be the pedal triangle of any point P.

We have

$$\overline{P_2P_3} = \overline{B_1P}\sin B_1 = \overline{B_1P}\,\frac{\sqrt{3}}{2}$$

$$\overline{P_2P_3}^2 + \overline{P_3P_1}^2 + \overline{P_1P_2}^2 = \tfrac{3}{4}(\overline{B_1P}^2 + \overline{B_2P}^2 + \overline{B_3P}^2).$$

But by a well known theorem,

$$\overline{B_1P}^2 + \overline{B_2P}^2 + \overline{B_3P}^2 = (\overline{B_1O}^2 + \overline{B_2O}^2 + \overline{B_3O}^2) + 3\overline{OP}^2 = b^2 + 3\overline{OP}^2$$

so that finally

$$\overline{P_2P_3}^2 + \overline{P_3P_1}^2 + \overline{P_1P_2}^2 = \tfrac{3}{4}b^2 + \tfrac{3}{4}\overline{OP}^2$$

and we see that if P moves on a circle about O, the sum of the squares of the sides of its pedal triangle is constant. Also it is well known that in the same case, the area of the pedal triangle is constant, and since

$$\cot\omega = \frac{\overline{P_2P_3}^2 + \overline{P_3P_1}^2 + \overline{P_1P_2}^2}{4\Delta}$$

the theorem is proved.

THEOREM 11. (SCHOUTE'S theorem) *In any triangle, the locus of a point whose pedal triangle has a constant Brocard angle is a circle coaxial with the circumcircle and the Brocard circle.*

The proof is effected by performing the inversion of theorem 9, noting theorem 7, and applying Theorem 10.

COROLLARY. *In particular, the locus of a point whose pedal triangle has the same Brocard angle (in the same sense) as the given triangle is the Brocard circle; and for the same Brocard angle but with the triangle described in the opposite sense, the locus is the Lemoine line.*

For, in the first place, the Brocard circle contains a point O, whose pedal triangle is similar to the given triangle; and further, the points of this circle and of the Lemoine line are mutually inverse with regard to the circumcircle.

All previous proofs by elementary methods, even of the corollary, have made use of a very awkward method of vertical projections from one plane to another (cf. Gallatly, l. c., or Emmerich, Die Brocard'sche Gebilde, §§ 60, 61). It will be noted that most of the steps of the present proof may fairly be regarded as general theorems, very little special proof being required for the theorem itself.

A SIMPLE RELATION BETWEEN ELEMENTARY NUMBER-THEORY AND ELEMENTARY PROJECTIVE GEOMETRY.

By AUBREY J. KEMPNER, Urbana, Ill.

In *L'Enseignement Mathématique*, 1916, p. 332, Mr. A. Reymond has published a note in which, with slight modifications, the following simple graphical method of determining whether a given integer is a prime number, is explained.

Write in a vertical and in a horizontal row the numbers 0, 1, 2, 3, 4, $\cdots$, as

0	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	2								
3	1		3							
4	1	2		4						
5	1				5					
6	1	2	3			6				
7	1						7			
8	1	2		4				8		
9	1		3						9	
10	1	2			5					10

FIG. 1.

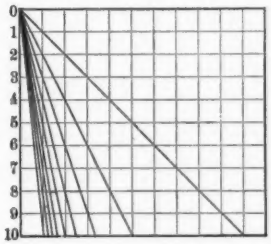


FIG. 2.

far as one likes, and fill out Fig. 1 in the way indicated. Evidently every number in our first vertical row, for example 6, will have for factors just the numbers which are in the same horizontal row, in this case 1, 2, 3, 6. Here the number 1 and the number 6 are both considered factors of 6. The number 7 has only the

two factors 1 and 7 and is therefore prime. In general, a number is prime when, and only when, in the same horizontal row with it, there are exactly two numbers. (The number 0, which contains every integer $\neq 0$ as a factor, and the number 1, are exceptional, as can be read off from Fig. 1.)

The method by which Fig. 2 is derived from Fig. 1 is obvious and requires no explanation. If we call the points in Fig. 2, in which exactly three straight lines intersect, knot-points, for brevity, then we see that a number is a prime number when and only when the horizontal line belonging to it in the figure has exactly two knot-points (excepting the special numbers 0, 1). In general, the number of distinct divisors of n (including 1 and n among the divisors) is exactly equal to the number of knot-points lying on the corresponding horizontal line.

For practical purposes, the construction is without value (and this quite apart from the fact that its application to any but small numbers becomes impossible on account of the complexity of the figure), because we have no means of deciding, in an empirical drawing, whether three lines meet in a point, or in three points forming the vertices of a small triangle. Also, as Mr. Reymond points out, it seems unlikely that results of theoretical interest may be obtained by a closer study of the construction (which is essentially a graphical interpretation of the method known under the name "Sieve of Eratosthenes." See any book on elementary number-theory). In spite of these objections, the construction is of interest as bringing out clearly the fact that the problem of deciding whether a given integer is a prime number, may be graphically solved by means of ruler and compasses. When one remembers, however, that only rational operations are required in order to determine the prime character of a number n (trial-divisions by integers less than n; more precisely trial-divisions by all prime numbers equal to or less than the square root of n) and even to determine all of its factors, it is clear that a linear construction with straight-edge[1] alone should be possible.

For justification of this statement, in case it should not be evident, see, for example, Weber-Wellstein, referred to below, pp. 210, 211.

To obtain such a construction, we shall only have to construct a figure perspective to Fig. 2. For readers not familiar with the idea of a projective scale, an explanation and references are given.

I. *Projective Scales.*[2] Consider an ordinary metric scale with a set of equidistant points representing for example the positive integers and 0. We wish

[1] Ruler without scale and without marks of any kind; an instrument used solely for drawing straight lines.

[2] See for example: YOUNG, "The theory of sets of points," 1906, p. 9, or WEBER-WELLSTEIN, "Encyklopädie der Elementar-Mathematik," Vol. II, 2d ed., 1907, p. 198. STEINER ("Die geometrischen Constructionen ausgeführt mittelst der geraden Linie und eines festen Kreises," Berlin, 1833. See "Gesammelte Werke," Vol. I, p. 499, or "Ostwald's Klassiker, Nr. 60," Chapter 3) showed that it is possible to carry out all constructions which may be made by use of straight-edge and compasses (equivalent to graded ruler and compass, provided a fixed unit is given) by means of a straight-edge alone, provided one fixed circle and its center are given. D. CAUER (*Mathematische Annalen*, 1912–13, Vol. 73, p. 90) proved that the center of the circle cannot be dispensed with.

to construct an equivalent projective scale. Introducing on our ordinary scale the point at infinity (∞) as representing the ∞ of real numbers (where no distinction is made between $+\infty$ and $-\infty$, the assumption being made in projective geometry that each straight line contains *one* point at infinity), we may interpret our metric scale in the following manner:
Forming the cross-ratio

$$(a_1, a_2, a_3, \infty) = \frac{a_1 - a_2}{a_3 - a_2} : \frac{a_1 - \infty}{a_3 - \infty},$$

we find

$$(a_1 a_2 a_3 \infty) = -1,$$

when a_1, a_2, a_3 are any three equidistant points, that is, when $a_3 - a_2 = a_2 - a_1$, since $(a_1 - \infty)/(a_3 - \infty)$ must be counted equal to unity, as one sees by a simple limiting process. Hence the four points a_1, a_2, a_3, ∞, taken in this order, are harmonic points. We make for the construction of a projective scale the following assumptions. Instead of the point at infinity on our metric scale, we choose on our line an arbitrary point u, which shall play on our projective scale the rôle of the point at infinity on our ordinary scale; we accordingly agree to consider three points a_1, a_2, a_3 as equidistant, that is, $a_3 - a_2 = a_2 - a_1$, when the four points a_1, a_2, a_3, u are harmonic points. By repeatedly using the classical construction for the determination of the fourth harmonic point when three points are given we see the truth of the following theorem:

When we are given on a line three points, a_1, a_2, u, which shall represent, respectively, 0, 1, ∞, then it is possible to construct as far as one likes by means of straight-edge alone[1] a projective scale containing the points representing 0, 1, 2, 3, 4, 5, $\cdots$.[2] The construction of the scale is unique, when 0, 1, ∞ are arbitrarily given; this fact may also be expressed by saying that a projective scale is completely determined by three of its points.

It is also possible to obtain, by straight-edge construction, a projective scale containing as many "rational" points as one likes, but we shall need only the points corresponding to integers. Likewise, it is beyond our purposes to introduce the "negative" numbers of the projective scale. For all this, and for the important question as to what assumptions must be made to ensure continuity in the projective plane, see either Young or Weber-Wellstein, referred to above.

For measurements in projective geometry, these projective scales usually take the place of ordinary scales in metric geometry. The ordinary metric scale is obviously a very special projective scale, obtained by assuming u at infinity and taking the length $a_1 \cdots a_2$ for the unit of length. Note that if 0, 1, 2, 3, 4, $\cdots$, form a projective scale, then also 0, 2, 4, 6, 8, $\cdots$ form a projective scale,

[1] For the construction of the ordinary metric scale, where u is actually the point at infinity on the line, compasses cannot be avoided. Compare, however, the reference to Steiner given above.

[2] We shall generally call these points simply 0, 1, 2, 3, $\cdots$, or 0′, 1′, 2′, 3′, $\cdots$, or 0″, 1″, 2″, 3″, $\cdots$, while the point u will be denoted by ∞, or ∞', or ∞''.

and in general $0, k, 2k, 3k, 4k, \cdots$ form a similar scale. (For our purposes k may be assumed a positive integer.)

II. We make now the following construction (compare Fig. 3):

In any triangle ABC, construct on BC a projective scale, taking $B = 0$, $C = \infty$, and an arbitrary point between B and C for 1. Similarly construct on

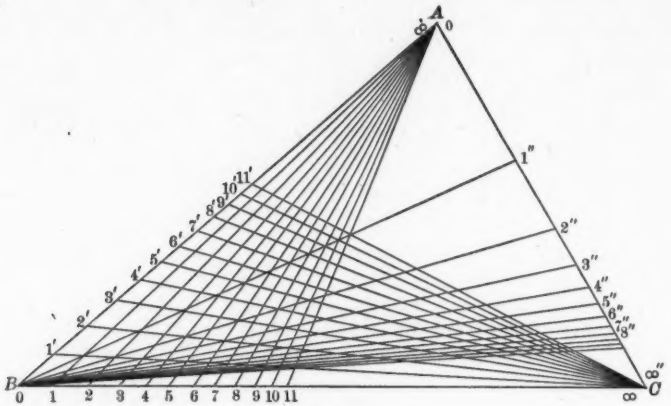


FIG. 3.

AB a projective scale, taking $B = 0$, $A = \infty$, and an arbitrary point between B and A for 1. The points on the scale BC may be called 0, 1, 2, 3, 4, $\cdots$, the points on the scale AB, $0'$, $1'$, $2'$, $3'$, $4'$, $\cdots$. Assume the rays $A1, A2, A3, A4, \cdots$ drawn, and also the rays $C1', C2', C3', C4', \cdots$. The point of intersection of two rays An, Cm' may be denoted by (n, m'). Then it follows from fundamental theorems of projective geometry that the points $(0, 0') = B$, $(1, 1')$, $(2, 2')$, $(3, 3')$, $(4, 4')$, $\cdots$ all lie in a straight line l_1. Taking into account the remark at the end of I, we see also that $(0, 0')$, $(2, 1')$, $(4, 2')$, $(6, 3')$, $(8, 4')$, $\cdots$ all lie in a straight line l_2, and in general $(0, 0')$, $(k, 1')$, $(2k, 2')$, $(3k, 3')$, $(4k, 4')$, $\cdots$, (k any positive integer) all lie in a straight line l_k. These lines $l_1, l_2, l_3, \cdots l_k \cdots$ all pass through B; denoting the intersection of l_k and AC by k'', it is now easy to see that, provided we call $A = 0''$[1] and call $C = \infty''$, the points $0''$, $1''$, $2''$, $3''$, $4''$, $\cdots$ on AC also form a projective scale.

III. From what has been said, we may derive the following result (see Fig. 3):

Assume any triangle ABC, choose on BC any point 1, and construct the projective scale $0, 1, 2, 3, \cdots$ on BC, as far as one likes, taking $B = 0$ and $C = \infty$; similarly, construct the projective scale $0', 1', 2', 3', \cdots$ on BA, choosing any point $1'$ on AB, and taking $B = 0'$, $A = \infty'$. Draw the rays $A1, A2, \cdots An \cdots$ and $C1', C2', \cdots Cm' \cdots$. Call the intersection of An and Cm' (n, m'). Draw

[1] This does not contradict our assumption, made above, that A represent ∞', since we were then considering A as a point on AB, while we are now considering A as a point on AC. The two determinations are independent of each other.

the ray $B(1, 1')$. This ray intersects AC in a point which we call $1''$. Construct the projective scale $0''$, $1''$, $2''$, $3''$, $4''$, $\cdots$ on AC, taking $A = 0''$, $C = \infty''$, and draw the rays $B2''$, $B3''$, $B4''$, $\cdots$. We thus have in our triangle a set of rays from each vertex.[1] Mark in the figure all points in which three rays meet. These points we shall call, for convenience, knot-points. By comparing Fig. 3 with Fig. 1 and Fig. 2, we find:

The number r of knot-points on An, $n = 2, 3, 4, 5, \cdots$, gives the number of distinct divisors of n, counting both 1 and n as divisors. Therefore n is prime, when and only when there are two knot-points on An. When $r = 3$, n is the square of a prime number, and, in a similar way, some other elementary results (for example the determination of common prime factors of given integers) may be read off the figure.—Readers familiar with projective geometry will recognize immediately how Fig. 3 may be derived from Fig. 2.

Other projective geometric constructions for determining the prime-character of a given number are easily devised, but the construction given above has the advantage of involving only straight lines.

When one ignores simple relations like the one treated in this note, between projective geometry and number theory, which in reality are only analogies without deeper import, the problem of finding connections between projective geometry and number theory seems to be, according to an interesting remark by Klein (Elliptische Modulfunktionen, Vol. I, p. 242, footnote), a task of great difficulty. (See also Weber-Wellstein, II, 1907, pp. 211–212, footnote.)

CONCERNING PREFERENTIAL VOTING.

By L. L. DINES, University of Saskatchewan.

In a recent number of the MONTHLY, Professor W. V. Lovitt has given an interesting discussion of a problem which may be stated as follows:

In a certain election, each voter expresses his first, second, and third choice. The three candidates A, B, and C receive respectively A_i, B_i, and C_i votes for ith choice. It is required to determine what weights may be assigned to the votes for first, second, and third choice in order that A may win.

While the purely algebraic discussion given in the present note is less vivid than the geometric discussion of Professor Lovitt, it may be of interest as it leads more directly to the necessary conditions for the existence of a solution, and to explicit ranges of possible values for the weights in terms of the data A_i, B_i, and C_i. Furthermore, the algebraic method can be extended to a treatment of the more general problem in which there are n candidates, and each voter

[1] The $\triangle ABC$ now represents, for purposes of projective measurement, the whole first quadrant in an ordinary system of coördinates. It is interesting to see how, by taking on the sides of our triangle (extending each indefinitely in both directions) the "negative" and the "rational" points into account, the whole ordinary plane is covered by a "projective" plane.

expresses his first, second, $\cdots$, and mth choice. This more general problem is considered briefly in § 2.

§ 1. Let the weights of first, second, and third choice be x_1, x_2, and x_3 respectively, with the condition

$$x_1 > x_2 > x_3 > 0. \tag{1}$$

Then since the number of points received by A is given by $A_1x_1 + A_2x_2 + A_3x_3$, with similar expressions for the number of points received by B and C, it follows that the necessary and sufficient conditions that A shall win are expressed by the inequalities

$$\begin{aligned}(A_1 - B_1)x_1 + (A_2 - B_2)x_2 + (A_3 - B_3)x_3 > 0,\\ (A_1 - C_1)x_1 + (A_2 - C_2)x_2 + (A_3 - C_3)x_3 > 0.\end{aligned} \tag{2}$$

These inequalities may be simplified by introducing instead of the weights, the differences between the weights, by the substitutions

$$x_1 - x_2 = \xi_1, \quad x_2 - x_3 = \xi_2, \quad x_3 = \xi_3,$$

or

$$x_1 = \xi_1 + \xi_2 + \xi_3, \quad x_2 = \xi_2 + \xi_3, \quad x_3 = \xi_3,$$

with the condition (1) replaced by the conditions

$$\xi_i > 0 \qquad i = 1, 2, 3. \tag{1'}$$

Expressed in terms of the ξ's, conditions (2) are

$$(A_1 - B_1)(\xi_1 + \xi_2 + \xi_3) + (A_2 - B_2)(\xi_2 + \xi_3) + (A_3 - B_3)\xi_3 > 0,$$

$$(A_1 - C_1)(\xi_1 + \xi_2 + \xi_3) + (A_2 - C_2)(\xi_2 + \xi_3) + (A_3 - C_3)\xi_3 > 0;$$

which upon collecting terms in ξ_1, ξ_2, and ξ_3, and making use of the relations

$$A_1 + A_2 + A_3 = B_1 + B_2 + B_3 = C_1 + C_2 + C_3 = \text{number of voters}, \tag{3}$$

may be written

$$(A_1 - B_1)\xi_1 > (A_3 - B_3)\xi_2 \tag{4}$$

$$(A_1 - C_1)\xi_1 > (A_3 - C_3)\xi_2. \tag{5}$$

Since ξ_3 does not appear in these condition, ξ_3 $(= x_3)$ may have any positive value. Also since (4) and (5) obviously impose a restriction only on the ratio $\xi_1 : \xi_2$, ξ_2 $(= x_2 - x_3)$ may have any positive value. It remains to determine the ranges of values which ξ_1 may have when conditioned by (4) and (5). The nature of these conditions is made more apparent by separating them into cases according to the value of the coefficient of ξ_1, as follows:

(4′) (*a*) If $A_1 - B_1 > 0$, $\xi_1 > \dfrac{A_3 - B_3}{A_1 - B_1}\xi_2$; (*b*) If $A_1 - B_1 = 0$, $(A_3 - B_3)\xi_2 < 0$;

(*c*) If $A_1 - B_1 < 0$, $\xi_1 < \dfrac{A_3 - B_3}{A_1 - B_1}\xi_2$;

(5′) (*a*) If $A_1 - C_1 > 0$, $\xi_1 > \dfrac{A_3 - C_3}{A_1 - C_1}\xi_2$; (*b*) If $A_1 - C_1 = 0$, $(A_3 - C_3)\xi_2 < 0$;

(*c*) If $A_1 - C_1 < 0$, $\xi_1 < \dfrac{A_3 - C_3}{A_1 - C_1}\xi_2$.

In considering the possible order relations which may exist between the three quantities A_1, B_1, C_1, those which differ from each other only in having B_1 and C_1 interchanged may for our purposes be considered equivalent. With this understanding, the following table which can now be filled out by reference to the conditions (4′), (5′), and (1′), gives the complete solution of our problem for all cases.

Case	Necessary Conditions	Range for ξ_1
$A_1 > B_1 \geqq C_1$	.....	$\xi_1 > \left\{\dfrac{A_2 - B_2}{A_1 - B_1}\xi_2 \text{ and } \dfrac{A_3 - C_3}{A_1 - C_1}\xi_2\right\}$
$A_1 = B_1 > C_1$	$A_3 < B_3$*	$\xi_1 > \dfrac{A_3 - C_3}{A_1 - C_1}\xi_2$
$A_1 = B_1 = C_1$	$A_3 < B_3$*, $A_3 < C_3$*	ξ_1 any positive value
$C_1 > A_1 > B_1$	$\dfrac{A_3 - B_3}{A_1 - B_1} < \dfrac{A_3 - C_3}{A_1 - C_1}$, $A_2 < C_2$	$\dfrac{A_3 - B_3}{A_1 - B_1}\xi_2 < \xi_1 < \dfrac{A_3 - C_3}{A_1 - C_1}\xi_2$
$C_1 > A_1 = B_1$	$A_3 < B_3$*, $A_3 < C_3$	$\xi_1 < \dfrac{A_3 - C_3}{A_1 - C_1}\xi_2$
$B_1 \geqq C_1 > A_1$	$A_3 < B_3$, $A_3 < C_3$	$\xi_1 < \left\{\dfrac{A_2 - B_2}{A_1 - B_1}\xi_2 \text{ and } \dfrac{A_3 - C_3}{A_1 - C_1}\xi_2\right\}$

It may be noted that on account of the relations (3), the conditions $A_3 < B_3$ and $A_3 < C_3$ when marked with a * in the column of necessary conditions, may be replaced respectively by $A_2 > B_2$ and $A_2 > C_2$.

§ 2. Suppose now there are $n + 1$ candidates: A, and $B^{(i)}$ $(i = 1, 2, \cdots, n)$; and the jth choice is to receive the weight x_j $(j = 1, 2, \cdots, m)$, with the condition $x_k > x_{k+1} > 0$. Then the necessary and sufficient conditions that A be elected are

$$\sum_{j=1}^{m} (A_j - B_j^{(i)})x_j > 0, \qquad i = 1, 2, \cdots, n. \tag{6}$$

If we now introduce as new variables the differences between the weights, by means of the substitutions

$$x_j = \sum_{h=j}^{m} \xi_h, \qquad j = 1, 2, \cdots, m,$$

we obtain in place of (6) the conditions

$$\sum_{j=1}^{m} (A_j - B_j^{(i)}) \sum_{h=j}^{m} \xi_h > 0, \qquad i = 1, 2, \cdots, n,$$

which may be written

$$\sum_{h=1}^{m}\sum_{j=1}^{h}(A_j - B_j^{(i)})\xi_h > 0, \qquad i = 1, 2, \cdots, n. \tag{7}$$

The n conditions of this set may be divided into three sets, according as $A_1 - B_1^{(i)}$, the coefficient of ξ_1, is positive, zero, or negative. Without loss of generality, we may suppose that in the first n_1 inequalities of (7) the leading coefficient is positive, in the next n_2 inequalities it is zero, while in the remaining n_3 $(= n - n_1 - n_2)$ inequalities it is negative. Conditions (7) are then replaced by the following three sets:

Type I, in which $A_1 - B_1^{(i)} > 0$, by

$$\xi_1 > -\sum_{h=2}^{m}\sum_{j=1}^{h}\frac{A_j - B_j^{(i)}}{A_1 - B_1^{(i)}}\xi_h, \qquad i = 1, 2, \cdots, n_1; \tag{I}$$

Type II, in which $A_1 - B_1^{(i)} = 0$, by

$$\sum_{h=2}^{m}\sum_{j=1}^{h}(A_j - B_j^{(i)})\xi_h > 0, \quad i = n_1 + 1, n_1 + 2, \cdots, n_1 + n_2; \tag{II}$$

Type III, in which $A_1 - B_1^{(i)} < 0$, by

$$\xi_1 < -\sum_{h=2}^{m}\sum_{j=1}^{h}\frac{A_j - B_j^{(i)}}{A_1 - B_1^{(i)}}\xi_h, \quad i = n_1 + n_2 + 1, n_1 + n_2 + 2, \cdots, n. \tag{III}$$

Of these three sets it will be noticed that (II) places no restriction on ξ_1, while (I) and (III) state respectively n_1 lower bounds and n_3 upper bounds for ξ_1. Since these upper bounds must all be positive, $\xi_2, \xi_3, \cdots, \xi_m$ must satisfy the n_3 conditions

$$\sum_{h=2}^{m}\sum_{j=1}^{h}(A_j - B_j^{(i)})\xi_h > 0, \quad i = n_1 + n_2 + 1, n_1 + n_2 + 2, \cdots, n. \tag{8}$$

Also in order that there may be an existent range for ξ_1 between the lower bounds of (I) and the upper bounds of (III), the quantities $\xi_2, \xi_3, \cdots, \xi_m$ must also satisfy the $n_1 n_3$ conditions

$$\sum_{h=2}^{m}\sum_{j=1}^{h}\left(\frac{A_j - B_j^{(i)}}{A_1 - B_1^{(i)}} - \frac{A_j - B_j^{(k)}}{A_1 - B_1^{(k)}}\right)\xi_h > 0, \tag{9}$$

$$i = 1, 2, \cdots, n_1,$$

$$k = n_1 + n_2 + 1, n_1 + n_2 + 2, \cdots, n.$$

We now have ξ_1 restricted only by (I) and (III); and the existence of a positive range for ξ_1 satisfying these conditions is assured by (8) and (9). The next step is to consider the conditions imposed upon $\xi_2, \xi_3, \cdots, \xi_m$ by (II), (8) and (9). These form a system of linear, homogeneous inequalities ($n_2 + n_3 + n_1 n_3$ in number), which may now be classified into Types I, II and III, according as the coefficients of ξ_2 are positive, zero, or negative. Those of Types I and III will

determine certain lower and upper bounds for ξ_2, while those of Type II, together with the conditions that the range for ξ_2 shall contain positive values, will constitute a system of linear, homogeneous inequalities in $\xi_3, \xi_4, \cdots, \xi_m$.

The process indicated can now be repeated, the ranges for $\xi_3, \xi_4, \cdots$ being determined successively, each in terms of the ξ's of higher subscript. After the first $(r-1)$ ξ's have been thus *eliminated*, we are met by a system of conditions of form

$$\sum_{h=r}^{m} a_h^{(l)} \xi_h > 0, \qquad l = 1, 2, \cdots, l_r, \tag{10}$$

where the coefficients $a_h^{(l)}$ are rational functions of the differences $A_j - B_j^{(i)}$ This system (10) may be classified into Types I, II, and III, according as the leading coefficients $a_r^{(l)}$ are positive, zero, or negative.

A necessary and sufficient condition that there exist a system of weighting under which A wins, is that at some stage of the process of successive elimination of the variables ξ_i described above, the inequalities of the system (10) *presenting itself shall be all of Type I.*

To show the necessity of the condition, we suppose that the system of conditions presenting itself at each stage of the elimination contains inequalities of Types II or III. Since the elimination of the ξ of lowest subscript from such a system leads to a system of conditions upon the ξ's of higher subscript, it follows that under our assumption, the successive elimination of the ξ's will lead finally to a system of conditions

$$a_m^{(l)} \xi_m > 0, \quad l = 1, 2, \cdots, l_m,$$

which contains only ξ_m. But according to our assumption not all the coefficients $a_m^{(l)}$ are positive. Hence the system of conditions cannot be satisfied by a positive ξ_m.

If, on the contrary, the inequalities (10) at any stage are all of Type I, they determine only *lower* bounds for ξ_r. The variables $\xi_{r+1}, \xi_{r+1}, \cdots, \xi_m$ can then have any positive values, ξ_r can have any positive values greater than these lower bounds, and positive values can be assigned to $\xi_{r-1}, \xi_{r-2}, \cdots, \xi_1$, in order, satisfying the conditions imposed in the elimination process.

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

The Continuum and Other Types of Serial Order. By EDWARD V. HUNTINGTON. Second Edition. Harvard University Press, Cambridge, Mass., 1917.

It is to some extent customary for younger students of our subject to cultivate skill in the use of great mathematical doctrines without perceiving the logical setting, in mathematical and physical science, of the complex and real number systems upon which most of these doctrines depend. And, indeed, we can all

recall a time in our studies when "rigorous foundations," kindly supplied by Dedekind and Cantor for the arithmetic theory of irrational numbers, were hardly short of puzzling. Our admiration for this beautiful theory was tempered if not attenuated by our inability to discern the *raison d'être* of such constructions. Everywhere in the current texts we found, to be sure, an explicit formulation of the fundamental idea that mathematics, as a scientific method, is the postulational theory of abstract systems; and that its relation to other branches of science consists in the possibility of applying these theories to the elucidation of mechanical, geometrical, or other systems presented in experience; an application conditioned solely on the validity of postulates, and the existence of one-to-one correspondences preserving certain relations. But foundational ideas meant for us little more than they appear to mean for a few professional philosophers who frequently publish lengthy criticisms of mathematical method. We were unacquainted with studies made by Veronese, Hilbert, Levi-Cività and others in the foundations of geometry for ordinary space; and evidence was overwhelming that the number system and the "continuous" line of points which we had used interchangeably in analytic geometry were ordinally and otherwise equivalent and inseparable. In fact we should not have regarded as quite sane the conception that other number systems of radically different ordinal properties might not be disqualified as bases for the mathematical description of our intuitional "continuum." We did not suppose that there might be limitations on the minuteness of the analysis to which the content of intuitional notions could be subjected in consciousness. In introducing beginners to the simple postulational science of a system with order relation, Huntington at once chooses the best means of exemplifying general ideas, and provides an explanation, intelligible to all, of the delicate and difficult real number concept.

Naturally no attempt is made in eighty short pages to familiarize readers with every important or interesting result in the theory of serial order types. A great body of striking relations and theorems given or described in the papers of Hausdorff, König and Hessenberg, in the Schoenfliess Bericht, and elsewhere, is quite properly omitted. But ample material is provided, both in this theory and in the more general *Mengenlehre*, to serve as a foundation for further profitable reading; and excellent indications are given to guide the student in such reading. The cardinal comparison and the potencies of general classes are discussed sufficiently to bring out the most significant problems in the doctrine of equivalence; cardinal numbers and their relation to potencies as well as to finite and transfinite ordinal numbers are clearly defined. Much is here gained in coherency of exposition by the simple account which is given of Hartog's 1915 theorem on well-ordering of classes.

Of particular interest to the reviewer were two wholly unrelated features of Huntington's "Continuum." In the first place the book is professedly and undeniably available for use in non-mathematical circles without suffering in scientific tone; simplicity and attractiveness are somehow made compatible with painstaking attention to details of argumentation. In the second place,

the treatment of continuously ordered series, with the emphasis placed on such types as possess an "n-dimensional framework" is noteworthy. It will stimulate better students to reflect on the significance, for any analysis based on a continuously ordered number system, of the presence or absence in that system of segments which are "denumerably framed." The relation of the classical analysis and function theory, which repose ultimately on the system of real numbers, to the distinctive presence in that system of denumerable dense sub-classes will be suggested; a relation responsible for all properties dependent upon the Generalized Heine-Borel Theorem for instance. For a similar situation outside of order theory, the student may consult Hausdorff's fascinating chapters on the "topological" (including the metric) geometries of general classes (Mengen) in his *Grundzüge der Mengenlehre*.

The first edition of the book under review appeared in 1905 as a reprint from two of the author's papers in the *Annals of Mathematics*. In the present edition (1917) the changes made include only a few modifications and additions. The account given of Hartog's paper is of course new. See Discussions, page 345, of this issue.

LESTER S. HILL.

PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Mo.

ALGEBRA.

485. Proposed by J. WALSH, Madison, Wisconsin.

Is it true that to every convergent series of positive terms, $a_1 + a_2 + a_3 + \cdots$, there corresponds a series of the type

$$\frac{M}{1^p} + \frac{M}{2^p} + \frac{M}{3^p} + \cdots, \qquad \text{such that} \qquad \frac{M}{k^p} > a_k, \quad p > 1?$$

486. Proposed by FLORENCE P. LEWIS, Goucher College, Baltimore, Md.

Find the condition which must be satisfied by the coefficients of the quartic,

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$$

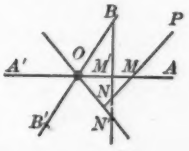
in order that the equation may be solvable by successive applications of the quadratic formula.

GEOMETRY.

518. Proposed by ROGER A. JOHNSON, Cleveland, Ohio.

If one angle of a triangle is 60°, the Euler line (the line through the circumcenter, orthocenter, and median point) is perpendicular to the bisector of that angle; and if one angle is 120°, the Euler line is parallel to the bisector of that angle.

519. Proposed by OTTO DUNKEL, Washington University.



Given the conjugate axes $A'OA$ and $B'OB$ of an ellipse, points of the curve may be constructed as follows: Drop the perpendicular BM' to OA and produce it to N' so that $BN' = AO$. Draw a straight line through O and N'. Upon a straight edge, say that of a slip of paper, the points N, M and P are marked so that $NM = N'M'$ and $MP = M'B$. Place the straight edge so that N falls on ON' and M on OM' and mark the position of P. This gives a point of the ellipse and by sliding the straight edge into new positions other points may be rapidly obtained. If the axes are perpendicular this gives the familiar trammel construction. Prove the correctness of this construction.

CALCULUS.

433. Proposed by LOUIS O'SHAUGHNESSY, University of Pennsylvania.

Solve the differential equation,

$$\frac{d^{1/2}y}{dx^{1/2}} - \frac{y}{x} = 0.$$

434. Proposed by E. W. CHITTENDEN, Champaign, Ill.

Evaluate $\int_0^1 f(x)dx$ where $f(x)$ is defined by the formula $f(x) = \sum_{n=1}^{n=\infty} \frac{1}{n2}$, signum $(x - x_n)$, and x_n denotes the nth number in the series $\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \cdots$.

Note.—For a real number k, "signum k," denotes $+1, 0, -1$ according as $k >, =, < 0$.

MECHANICS.

350. Proposed by J. B. REYNOLDS, Lehigh University.

If an elastic tube filled with liquid under pressure doubles in length, in what ratio will the radius be increased?

351. Proposed by G. PAASWELL, New York City.

A transition curve is one such that its curvature varies directly with the distance measured along the curve from its point of zero curvature, that is, from the tangent. Its intrinsic equation is given by $da/ds = ks$, the constant being determined from the fact that for a given length of transition the final radius of curvature, *i. e.*, the radius of the circle into which the transition runs, is given together with the length of transition. In making a turnout from the transition curve there is as yet no direct way of computing the functions which would completely locate this turnout. In Figure 1, the point of switch is at B and the frog point at C. The angle F is given, termed the frog angle, and either the location of C or B is given, whence it is required to find either B or C and the radius of the turnout. Note that all these data must be referred finally to the center lines of the tracks and not to the individual rails. In attempting approximate solutions do not replace the transition by the cubic parabola as that is not always a good approximation. (See Crandall, *The Transition Curve* for the discussion of the properties of this curve.)

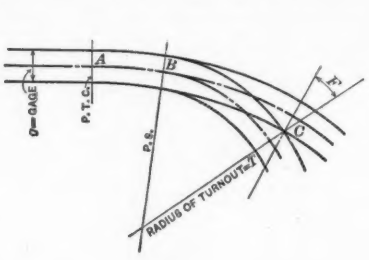


FIG. 1.

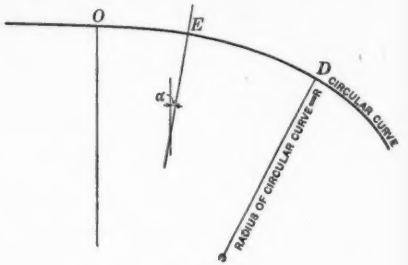


FIG. 2.

NUMBER THEORY.

268. Proposed by FRANK IRWIN, University of California.

Show that in any arithmetical progression, whose first term a_1 and common difference d are positive integers, any required number of consecutive terms may be found, no one of which is a prime number.

269. Proposed by ARTEMAS MARTIN, Washington, D. C.

Find three rectangular parallelepipedons whose edges are rational whole numbers, and their solid diagonals equal, and rational whole numbers.

SOLUTIONS OF PROBLEMS.

GEOMETRY.

504. Proposed by NATHAN ALTSHILLER, University of Oklahoma.

The base of a variable triangle is fixed, the opposite vertex describing a given line. Find the envelope of the side of the pedal triangle opposite the moving vertex.

SOLUTION BY SHELTON P. SANFORD, University of Georgia.

Let the base of the triangle be the line joining the points $(0, 0)$ and $(n, 0)$. Let the given line be (1) $y = mx + b$.

Then one side of the triangle will be (2) $y = ax$.

Solving these equations, we have $x = b/(a - m)$; $y = ab/(a - m)$.

This point of intersection and the point $(n, 0)$ determine the other side of the triangle, (3) $y = abx/H - abn/H$, where $H = b - an + mn$.

The altitude from the point $(n, 0)$ to the line $y = ax$ is (4) $y = -x/a + n/a$.

The altitude from the origin to the line $y = abx/H - abn/H$ is (5) $y = -Hx/ab$.

Solving (2) and (4), we have $x = n/(a^2 + 1)$; $y = an/(a^2 + 1)) \equiv P_1$.

Solving (3) and (5), we have

$$\left.\begin{aligned} x &= a^2b^2 \cdot n/(a^2b^2 + H^2) \\ y &= -abnH/(a^2b^2 + H^2) \end{aligned}\right\} \equiv P_2.$$

These two points of intersection (P_1 and P_2) are sufficient to determine the required side of the pedal triangle, which is found to be

$$y = ax(H + b)/(H - a^2b) - abn/(H - a^2b).$$

Substituting for H its value $(b - an + mn)$ and arranging according to powers of the parameter a, we have

$$a^2(nx - by) + a(bn - 2bx - mnx - ny) + (b + mn)y = 0.$$

Differentiating with respect to the parameter and eliminating, we have

$$x^2(4b^2 + 4bmn + m^2n^2) + y^2(4b^2 + 4bmn + n^2) - 2xymn^2 - x(4b^2n + 2mn^2) - 2ybn^2 + b^2n^2 = 0.$$

This equation represents a conic with one of its axes parallel to the given line ($y = mx + b$).

Also solved by PAUL CAPRON and OSCAR ADAMS.

505. Proposed by O. S. ADAMS, Coast and Geodetic Survey, Washington, D. C.

Construct a triangle, having given the sum of two sides, the angle included by these sides, and the altitude from the given angle upon the third side. (See figure on page 330.)

SOLUTION BY MRS. ELIZABETH B. DAVIS, U. S. Naval Observatory.

Let the sum of the two sides be $2k$, the included angle α, and the altitude on the third side R. Construct $\triangle BAC$, making $\angle A = \alpha$ and $AB = AC = k$. Draw AO, the bisector of $\angle A$, and erect perpendiculars to AB and AC at B and C, respectively, meeting AO in E. Then is E the focus of a parabola of which DO is the axis, and BC the tangent at the vertex D. Every tangent to this parabola cuts off on AB and AC intercepts whose sum is $AB + AC = 2k$. Therefore, the third side of the required $\triangle$ is a tangent to the parabola. Since the perpendicular on it from A is equal to R, it is also tangent to the circle whose center is A and radius R.

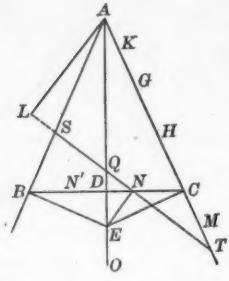
Therefore, the problem is to construct a common tangent to this parabola and circle.

On CA, lay off $CG = R$. Bisect CG at H, and with H as center and HE as radius describe a semicircle meeting AC, produced if necessary, in K and M. (It is sufficient for this purpose to find the points K and M.)

Then $GK = CM$. Also, E being on the circumference, and EC perpendicular to the diameter KM, we have

$$EC^2 = (R + GK) \cdot CM = (R + CM) \cdot CM.$$

From E as center, with radius CM, describe an arc intersecting BC in the points N and N'. Join EN (or EN'). Through N (or N'), draw ST (or $S'T'$), perpendicular, respectively to EN or EN'. Then $\triangle AST$ (or $A'S'T'$) is the required triangle.



Proof.—Draw AL (or AL') perpendicular to ST (or $S'T'$), produced if necessary. Since EN, the perpendicular from E to ST, meets ST in its intersection with BC, it follows that ST is tangent to the parabola. Therefore, the sum of its intercepts on AB and AC is $2k$. That is, $AS + AT = 2k$. Also $AL = R$. For, the $\triangle$'s ALQ and ENQ, both being right $\triangle$'s, and the vertical $\angle$'s at Q being equal, the $\triangle$'s are similar, and

$$AL : EN = AQ : EQ.$$

By composition,

$$(AL + EN) : EN = AE : EQ. \qquad (1)$$

But in the right $\triangle ENQ$, ND is perpendicular to the hypotenuse EQ. Hence, $EQ = \overline{EN}^2/ED$. Substituting this value of EQ in (1), and dividing the second and fourth terms by EN, we have

$$(AL + EN) : 1 = AE : EN/ED,$$

or

$$(AL + EN)\cdot EN = AE\cdot ED.$$

But in the right $\triangle ACE$, $AE\cdot ED = \overline{EC}^2$, and it has been shown that $\overline{EC}^2 = (R + CM)\cdot CM$. Therefore, $(AL + EN)\cdot EN = (R + CM)\cdot CM$. But $EN = CM$ by construction. Therefore $AL = R$.

As indicated in the proof, there are two possible solutions.

Also solved by OSCAR S. ADAMS and N. P. PANDYA.

506. Proposed by S. A. COREY, Albia, Iowa.

Given a pentagon, plane or gauche, whose sides a, b, c, d, e are represented by the vectors x, y, z, v and $(x + y + z + v)$, respectively; and a second pentagon whose sides a_1, b_1, c_1, d_1, e_1 are represented by the vectors r, s, t, u and $(r + s + t + u)$, respectively, where $r = c_1x - c_5c_2y - c_6c_3z + c_5c_6c_4v$, $s = c_2x + c_1y - c_6c_4z - c_6c_5v$, $t = c_1z + c_4x + c_5c_3v + c_5c_4y$, $u = c_1v - c_4x - c_2z + c_3y$; c_1, c_2, c_3, c_4, c_5, c_6, being ordinary scalars.

Then prove the existence of the following relation between the sides of the two pentagons:

$$(c_1^2 + c_5c_2^2 + c_6c_3^2 + c_5c_6c_4^2)(x^2 + c_5y^2 + c_6z^2 + c_5c_6v^2) = r^2 + c_5s^2 + c_6t^2 + c_5c_6u^2.$$

SOLUTION BY THE PROPOSER.

Inasmuch as the relation to be established between the vector sides is identical in form with a well-known algebraic identity, it is sufficient to show that the identity holds when certain of the algebraic (scalar) quantities are replaced by vectors. It is well known that the noncommutative character of vector multiplication does not affect the scalar part of the product of linear vector functions in the particular case where that product is a homogeneous *quadratic* function of the vectors employed. As this condition is satisfied in this problem the identity holds, and the existence of the given relation between the sides of the two pentagons is proved. (In forming the vector sides r, s, t, u the direction of the vectors x, y, z, v must be carefully noted.)

Inasmuch as vector *addition* is commutative it follows that the given pentagons may be replaced by certain other pentagons having the same vector sides placed in some other order of succession. Some of the sides may, of course, be zero, in which case the resulting figure would not be a pentagon. To illustrate, let x, y, z be the coterminous vector edges of any parallelepiped, v in this case being zero, and if $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 1$, then r, s, t and u are the diagonals connecting opposite corners of the parallelepiped, and the formula asserts that the sum of the squares of all its 12 edges equals the sum of the squares of its four diagonals; also that the shape of the parallelepiped may be changed without altering the sum of the squares of its diagonals, provided that no change is made in the length of its sides, or in the sum of their squares.

Similarly, if $c_1 = 0$, $v = 0$, $c_2 = c_3 = c_4 = c_5 = c_6 = 1$, the formula asserts that three times the sum of the squares of any three coterminous edges of any parallelepiped equals the sum of

the squares of the three diagonals formed by connecting the extremities of said edges plus the square of the diagonal of the parallelepiped originating at the intersection of said three edges.

Whether the formula could be used in determining the properties of figures in space of four dimensions depends on whether the noncommutative character of vector multiplication does or does not affect the scalar part of the product of linear vector functions, in such space, in the particular case where such product is a homogeneous quadratic function of the vectors employed. If applicable, the number of vectors employed in the formula, and its form, would indicate its usefulness in such higher space.

507. Proposed by A. A. BENNETT, University of Texas.

With the use of the compasses alone construct a circle with area five times as great as that of a given circle. (This problem is said to be due to Napoleon I.)

I. Solution by I. L. Miller, Indiana University.

This problem is equivalent to the construction of $R\sqrt{5}$, where R is the radius of the given circle.

Let A be the center of the given circle. (Fig. 1) Take any point B on its circumference and construct a circle of radius equal to that of the given circle. Let this circle intersect the given circle in C

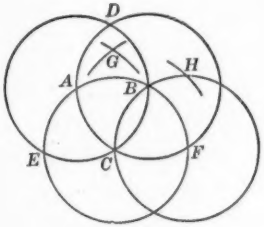


Fig. 1.

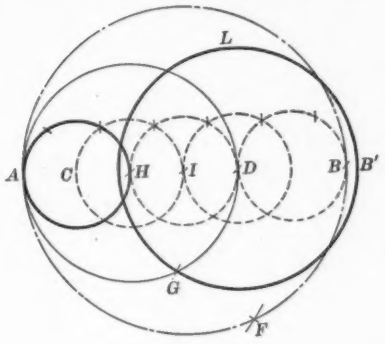


Fig. 2.

and D. Now with C as center construct another circle of radius R, intersecting the first two circles in E and F respectively. And finally with F as center construct another equal circle.

It is evident that $CD = R\sqrt{3}$. With E and F as centers and CD as radius strike arcs, intersecting in G. Then $CG = R\sqrt{2}$. With C as center and CG as radius strike an arc intersecting in H the circumference of the circle last drawn.

Then $EH = R\sqrt{5}$.

This also solves another problem due to Napoleon I; the quadrisection of the circumference of a circle by the use of the compasses alone, which is equivalent to the construction of $R\sqrt{2}$.

II. Solution by Gregory Breit, Student, Johns Hopkins University.

Let AC be the given circle, of radius r. (Fig. 2) From an arbitrary point A on its circumference, lay off three chords of radius length, reaching point H; HA is then a diameter of AC. With H as a center and radius r lay off another circle finding its diameter CI in the same manner as before. Continue this until five such circles are drawn with their centers in the line AB. With H as a center and $2r$ as a radius, lay off the circle AD. With I as center and $3r$ as radius lay off a circle AB. With a radius of length $5r$ cut this circle in F, using A as center. Describe an arc with radius BF from A as a center, cutting circle AD in G. With DG as radius and D as center describe the circle LGB' which is a circle whose area is five times the area of AC. For chord $AF = 5r$, and diameter $AB = 6r$. Hence, chord $BF = \sqrt{36r^2 - 25r^2} = r\sqrt{11}$, chord AG = chord $BF = r\sqrt{11}$, and diameter $AD = 4r$.

Hence, $DG = \sqrt{16r^2 - 11r^2} = r\sqrt{5}$. That is, circle LGB' is 5 times the given circle.

Also solved in various ways by J. W. BALDWIN, F. E. CANADAY, C. E. GITHENS, F. E. WOODS, J. E. HATCH, O. S. ADAMS, J. ROSENBAUM, J. M. STETSON, and J. E. MCMAHON, JR.

CALCULUS.

419. Proposed by C. C. YENN, Tangshan, North China.

Find the entire area of the surface $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$.

SOLUTION BY THE PROPOSER.

Using the formula $\iint \{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2\}^{1/2} dydx$, we have, from the equation of the surface,

$$\partial z/\partial x = -z^{1/3}/x^{1/3}, \qquad \partial z/\partial y = -z^{1/3}/y^{1/3};$$

whence

$$S = 8\int_0^a \int_0^{(a^{2/3}-x^{2/3})^{3/2}} \{(a^{2/3} - x^{2/3})x^{2/3} + (a^{2/3} - x^{2/3})y^{2/3} - y^{4/3}\}^{1/2} x^{-1/3} y^{-1/3} dydx. \tag{1}$$

To integrate with respect to y, let $y^2 = w^3$, so that $y^{-1/3}dy = \frac{3}{2}dw$, also put

$$A = (a^{2/3} - x^{2/3})x^{2/3}, \qquad B = (a^{2/3} - x^{2/3}). \tag{2}$$

Then

$$\int_0^{(a^{2/3}-x^{2/3})^{3/2}} \{(a^{2/3} - x^{2/3})x^{2/3} + (a^{2/3} - x^{2/3})y^{2/3} - y^{4/3}\}^{1/2} y^{-1/3} dy = \frac{3}{2}\int_0^B \{A + Bw - w^2\}^{1/2} dw$$

$$= \frac{3}{2}\left\{\frac{2w - B}{4}(A + Bw - w^2)^{1/2} + \frac{B^2 + 4A}{8}\sin^{-1}\left[\frac{2w - B}{(B^2 + 4A)^{1/2}}\right]\right\}_0^B.$$

Substituting the limits of integration, also the values of A and B from (2), we have from (1),

$$S = 6\int_0^a (a^{2/3} - x^{2/3})^{3/2} dx + 3\int_0^a (a^{2/3} - x^{2/3})(a^{2/3} + 3x^{2/3})x^{-1/3} \sin^{-1}\left\{\frac{a^{2/3} - x^{2/3}}{a^{2/3} + 3x^{2/3}}\right\}^{1/2} dx. \tag{3}$$

To evaluate the first integral, let $x = v^3$, $dx = 3v^2dv$, also put $a^{2/3} = \alpha^2$, then

$$6\int_0^a (a^{2/3} - x^{2/3})^{3/2} dx = 18\int_0^\alpha (\alpha^2 - v^2)^{3/2} v^2 dv$$

$$= -18\frac{v(\alpha^2 - v^2)^{5/2}}{6}\Big|_0^\alpha + 3\alpha^2\int_0^\alpha (\alpha^2 - v^2)^{3/2} dv$$

$$= 3\alpha^2\left\{\frac{v(\alpha^2 - v^2)^{1/2}(5\alpha^2 - 2v^2)}{8} + \frac{3\alpha^4}{8}\sin^{-1}\left(\frac{v}{\alpha}\right)\right\}_0^\alpha$$

$$= \frac{9}{8}\alpha^6\frac{\pi}{2} = \frac{9\pi a^2}{16}.$$

To evaluate the second integral, put $x^{2/3} = \lambda$, $\frac{3}{2}x^{-1/3}dx = d\lambda$, also set $a^{2/3} = k$, then the second term of S in (3) becomes

$$3 \times \frac{3}{2}\int_0^k (k - \lambda)(k + 3\lambda)\sin^{-1}\left\{\frac{k - \lambda}{k + 3\lambda}\right\}^{1/2} d\lambda$$

$$= \frac{9}{2}(k^2\lambda + k\lambda^2 - \lambda^3)\sin^{-1}\left\{\frac{k - \lambda}{k + 3\lambda}\right\}^{1/2}\Big|_0^k + \frac{9}{2}\int_0^k (k^2\lambda + k\lambda^2 - \lambda^3)\frac{kd\lambda}{(k + 3\lambda)\{\lambda(k - \lambda)\}^{1/2}}, \tag{4}$$

where the first term vanishes at both limits of integration. To integrate the second term, let $\sqrt{\lambda(k - \lambda)} = \lambda\xi$, then $\lambda = k/(1 + \xi^2)$, $d\lambda = -2k\xi d\xi/(1 + \xi^2)^2$; and the second term of (4) becomes

$$-9k^3\int_\infty^0 \left(\frac{1}{1 + \xi^2} + \frac{1}{(1 + \xi^2)^2} - \frac{1}{(1 + \xi^2)^3}\right)\frac{d\xi}{\xi^2 + 4} = 9k^3 \lim_{c=\infty}\int_0^c \frac{(\xi^4 + 3\xi^2 + 1)d\xi}{(1 + \xi^2)^3(\xi^2 + 4)}.$$

Separating into partial fractions, we have

$$\int \frac{(\xi^4+3\xi^2+1)d\xi}{(1+\xi^2)^3(\xi^2+4)} = \frac{5}{27}\int \frac{d\xi}{\xi^2+1} + \frac{4}{9}\int \frac{d\xi}{(\xi^2+1)^2} - \frac{1}{3}\int \frac{d\xi}{(\xi^2+1)^3} - \frac{5}{27}\int \frac{d\xi}{\xi^2+4}.$$

Applying the formula

$$\int \frac{d\xi}{(\xi^2+a^2)^n} = \frac{1}{2(n-1)a^2}\left\{\frac{\xi}{(\xi^2+a^2)^{n-1}} + (2n-3)\int \frac{d\xi}{(\xi^2+a^2)^{n-1}}\right\} \quad [\text{here } a=1],$$

to the second and third integrals repeatedly, simplifying and integrating, we get

$$\frac{7\xi}{72(\xi^2+1)} - \frac{\xi}{12(\xi^2+1)^2} + \frac{61}{216}\arctan\xi - \frac{5}{54}\arctan\frac{\xi}{2}.$$

Substituting the limits of integration, letting c approach ∞, and writing $a^{2/3}$ for k, we obtain

$$\frac{41}{48}\pi a^2$$

as the value of the second term of S in (3).

Hence, finally,

$$S = \frac{9\pi a^2}{16} + \frac{41\pi a^2}{48} = \frac{17}{12}\pi a^2,$$

which is the area required.

Note.—The value of S thus obtained is 34/9 [< 4] times the area of the projection of the surface on one of the coördinate planes; the latter area being $\frac{3}{4}\pi a^2$.

Also solved by E. F. CANADAY, PAUL CAPRON, and J. B. REYNOLDS.

421. Proposed by E. H. MOORE, The University of Chicago.

Given n continuous real-valued functions $\phi_g(x)$ $(g = 1, 2, \ldots, n)$ of the real variable x on the interval (01) and set $\exp \int_0^1 \phi_g(x)\phi_h(x) = w_{gh}$ $(g, h = 1, 2, \ldots, n)$. Prove that the determinant $|w_{gh}|$ of the matrix (w_{gh}) is always $\geqq 0$ and that it is > 0 if no two of the functions $\phi_1, \ldots, \phi_n$ are identically equal on (01).

SOLUTION BY C. F. GUMMER, Kingston, Ontario.

[*Note.* Since the appearance in the June issue of my "solution" for this problem. I have learned that it is based on a misinterpretation of the question. Owing probably to the uncommon use of the period in "exp." (as originally printed) I overlooked the meaning of this as a sign for the exponential function, and foolishly read it as an abbreviation for "expression." I wish to make this explanation of the error and to present my apologies to the proposer for doubts cast on the correctness of the problem. C. F. GUMMER].

Let C_{gh} denote $\int_0^1 \phi_g(x)\phi_h(x)dx,$

so that $w_{gh} = \exp c_{gh}$. The theorem will be generalized by putting in the place of the exponential function any function $F(z)$ of the form $\sum_{m=0}^{\infty} a_m z^m$ having a radius of convergence greater than every c_{gh} in absolute value and having the a's, for the first of the theorem positive or zero, and for the second part all positive. Let $W_{gh} = F(c_{gh})$

It was proved in the June number (in mistake for the present theorem) that the determinant $|c_{gh}| \geqq 0$. Since the same is true for the minors of (c_{gh}) coaxial with it, (c_{gh}) is the matrix of a positive or identically vanishing quadratic form, definite or semi-definite according as $|c_{gh}| > 0$ or $= 0$. This form (in variables $y_1, \ldots, y_n$) may be written

$$\sum_{i=1}^{r}\left(\sum_{j=1}^{n} b_{ij}y_j\right)^2,$$

the b's being real, and r being the rank of (c_{gh}).

$$\therefore \quad c_{gh} = \sum_{i=1}^{r} b_{ig} b_{ih}$$

$$\therefore \quad W_{gh} = \sum_{m=0}^{\infty} a_m \left(\sum_{i=1}^{r} b_{ig} b_{ih} \right) = \sum_{p_1, \cdots, p_r} \quad \underset{p_1, \cdots, p_r}{C} \left(b_{1g} b_{1h} \right)^{p_1} \left(b_{2g} b_{2h} \right)^{p_2} \left(b_{rg} b_{rh} \right)^{p_r},$$

$\underset{p_1, \cdots, p_r}{C}$ standing for

$$\frac{|\Sigma p}{\Pi | p} a_{\Sigma p}.$$

$\therefore$ (W_{gh}) is the matrix of the quadratic form

$$(1) \qquad \sum_{p_1, \cdots, p_r = 0, 1, \cdots} \quad \underset{p_1, \cdots, p_r}{C} \left(\sum_{j=1}^{n} b_{1j}^{p_1} \cdots b_{rj}^{p_r} y_j \right)^2,$$

which is also a positive or vanishing form, since the series in fact is absolutely convergent for arbitrary values of $y_1, \cdots, y_r$.

$\therefore |W_{gh}| \geqq 0$, which is the first part of the generalized theorem.

Suppose now that $|W_{gh}| = 0$, so that the form (1) is only semi-definite. The C's being now all positive, it follows that any form obtained from (1) by giving to the C's other positive values will be semi-definite. But $w_{gh^k} = \exp(kc_{gh})$, which differs from W_{gh} only in the values of the (positive) coefficients, k being positive. $\therefore (w_{gh^k})$ is the matrix of a semi-definite form. Therefore

$$(2) \qquad |w_{gh^k}| = 0, \qquad (k = 1, 2, \cdots).$$

Now every minor of $|w_{gh}|$ of the type $\begin{vmatrix} w_{gg} & w_{gh} \\ w_{gh} & w_{hh} \end{vmatrix}$ must be $\geqq 0$. If all such are > 0, then for every g and h $(g \neq h)$, w_{gh} has an absolute value $< \sqrt{w_{gg} w_{hh}}$. $\therefore$ the leading term of the determinant $|w_{gh}|$ is greater in absolute value than any other term and > 1. $\therefore \lim_{k=\infty} |w_{gh^k}| = \infty$, which contradicts (2). $\therefore$ for some g and h $(g \neq h)$, $w_{gg} w_{hh} = w_{gh}^2$.

That is $c_{gg} + c_{hh} = 2c_{gh}$, or $\int_0^1 (\phi_g - \phi_h)^2 dx = 0$. Hence, $\phi_g(x) = \phi_h(x)$ on (01).

422. Proposed by O. S. ADAMS, Coast and Geodetic Survey, Washington, D. C.

Prove that

$$\int_0^1 \int_0^1 f(xy)(1-x)^{m-1} y^m (1-y)^{n-1} dx dy = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \int_0^1 f(z)(1-z)^{m+n-1} dz,$$

$f(xy)$ being an arbitrary function of xy.

SOLUTION BY THE PROPOSER.

Let us arrange the integral as follows:

$$I = \int_0^1 dy \int_0^1 f(xy)(y - xy)^{m-1} (1-y)^{n-1} y dx,$$

the integration with respect to x to be made first, considering y as a constant.

Let $z = xy$, then $dz = y dx$. This gives us

$$I = \int_0^1 dy \int_0^y f(z)(y-z)^{m-1} (1-y)^{n-1} dz.$$

Changing the order of integrations, we have

$$I = \int_0^1 f(z) dz \int_z^1 (y-z)^{m-1} (1-y)^{n-1} dy.$$

For a justification of this step see Bôcher's *Integral Equations*, p. 4, Dirichlet's Formula.

Now let

$$y - z = (1 - z)w, \qquad dy = (1 - z)dw,$$

and

$$1 - y = (1 - z)(1 - w).$$

By substituting these results, we obtain

$$I = \int_0^1 f(z)(1-z)^{m+n-1}dz \int_0^1 w^{m-1}(1-w)^{n-1}dw = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \int_0^1 f(z)(1-z)^{m+n-1}dz.$$

This problem is taken from Whittaker & Watson's *Modern Analysis*, p. 250. Given in *Math. Trip.*, 1894.

Note.—A second solution was received but no name signed to it. EDITOR.

423. Proposed by J. B. REYNOLDS, Lehigh University.

Show that the envelope of all circles with their centers on the circle $x^2 + y^2 = a^2$ and tangent to the x-axis is the two-arched epicycloid.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

The general form of equation of circles fulfilling the conditions of the problem is

$$f(x, y, x_1, r) = (x - x_1)^2 + (y - r)^2 - r^2 = 0, \tag{1}$$

with the conditional equation

$$\varphi(x_1, r, a) = x_1^2 + r^2 - a^2 = 0. \tag{2}$$

We are to find the envelope of the system of circles represented by (1), x_1, r being the variable parameters. Using the undetermined multiplier λ and the auxiliary equations

$$\frac{df}{dx_1} = \lambda \frac{d\varphi}{dx_1}, \tag{3}$$

$$\frac{df}{dr} = \lambda \frac{d\varphi}{dr}, \tag{4}$$

we have

$$-(x - x_1) = \lambda x_1, \tag{5}$$

and

$$-(y - r) - r = \lambda r. \tag{6}$$

We must eliminate x_1, r, λ from (1), (2), (5), and (6). Substituting the values of $x - x_1$, and $y - r$ from (5) and (6) in (1),

$$\lambda^2 x_1^2 + r^2(1 + \lambda)^2 = r^2. \tag{7}$$

Also eliminating x_1 from (2) and (7),

$$\lambda(2r^2 + a^2\lambda) = 0. \tag{8}$$

The value of $\lambda = 0$ is irrelevant, but the second factor in (8) gives

$$r^2 = -\tfrac{1}{2}a^2\lambda. \tag{9}$$

Substituting (9) in (2) gives

$$x_1^2 = \frac{a^2}{2}(\lambda + 2) \tag{10}$$

and (5) gives

$$x_1 = \frac{x}{1 - \lambda}, \tag{11}$$

which with (10) gives the cubic in λ

$$a^2\lambda^3 - 3a^2\lambda + 2(a^2 - x^2) = 0. \tag{12}$$

Again, (6) gives

$$r = \frac{y}{\lambda}, \tag{13}$$

and this with (9) gives the second cubic in λ,

$$a^2\lambda^3 + 2y^2 = 0. \tag{14}$$

Eliminating λ from (12) and (14) by Sylvester's method, the required envelope is given by

$$4(x^2 + y^2 - a^2)^3 - 27a^4y^2 = 0. \tag{15}$$

This is a two-cusped epicycloid, as may be shown by eliminating θ from the two parametric equations of the epicycloid, a and b being the radii of the fixed and generating circles,

$$x = (a + b)\cos\theta - b\cos\frac{a+b}{b}\theta, \tag{16}$$

$$y = (a + b)\sin\theta - b\sin\frac{a+b}{b}\theta, \tag{17}$$

first supposing $b = \frac{1}{3}a$.

Note.—Since making this solution, I discovered that this problem was proposed by Sir Arthur Cayley as No. 1812 in the *London Educational Times*, and in the year 1852.

Cayley was one of the most cordial and active contributors to the mathematical section of the *Times*, and it is interesting to notice that his problems and the most of his solutions appearing in the monthly issues of that journal have been included in his *Works*.

Problem 1812 is reproduced there, but no solution is given. Reference, however, is made to problem 1771 and its solution, his statement being that he was led to No. 1771 by his study of 1812.

The statement of 1771 is: "Given a circle and a line, it is required a find a parabola, having the line for directrix, and the circle for its circle of curvature." The solution given is rather intuitional in character, justifying the equation of the parabola required by certain tests.

Employing rectangular axes, such that $x = m$ is the given line, and $x^2 + y^2 = 1$ the given circle, the required parabola is

$$y^2 - 2\left(1 - \frac{4}{9}m^2\right)^{3/2}y + \frac{16}{27}m^3x + 1 - \frac{4}{3}m^2 = 0 \tag{i}$$

and its focus

$$\left\{m - \frac{8}{27}m^3, \qquad \left(1 - \frac{4}{9}m^2\right)^{3/2}\right\}. \tag{ii}$$

Taking $(\cos\theta, \sin\theta)$ as coördinates of a variable point on the circumference of $x^2 + y^2 = 1$, Cayley merely states that

$$x = \frac{3}{2}\cos\theta - m\cos 2\theta + \frac{1}{2}\cos 3\theta,$$

$$y = \frac{3}{2}\sin\theta - m\sin 2\theta + \frac{1}{2}\sin 3\theta,$$

are the coördinates of a variable point on the required envelope, adding what is the interesting connection of Nos. 1812 and 1771, viz., the required envelope in 1812 is a curve of the sixth order and has two cusps, which are the *foci* in the result of solution of 1771. It is to be noticed that the unreduced form of Cayley's result shows that there are *two* parabolas.

By a simple transformation of axes it may be easily shown that the polar of (*ii*) with respect to (*i*) is $x = m$, as should be.

The only other place where I have seen our problem is in the American edition of Williamson's *Differential Calculus*, 1884, but there is no certainty about the date when the problem was assigned a place in the manuscript of that text.

Also solved by PAUL CAPRON, S. W. REAVES, H. S. BEERS, I. L. MILLER, WILLIAM WEBER, A. M. HARDING, HORACE OLSON, G. W. HARTWELL, C. P. SOUSLEY, R. A. JOHNSON, F. M. MORGAN, E. W. WORTHINGTON, D. F. BARROW, C. C. YEN, and the PROPOSER.

MECHANICS.

Problem 332 is the same as 490 in Geometry, the solution of which appeared in the February number of the MONTHLY.

333. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

A flywheel 21 feet in diameter makes 100 revolutions per minute. The weight of a cubic foot of its material is 448 pounds. Show that the intensity of stress on a transverse section of rim,

assuming that it is unaffected by the arms, is 1,176 lbs. per sq. in. If the safe stress permissible in the material is 6,000 lbs. per sq. in., show that the greatest speed at which the wheel can be run with safety is about 225 revolutions per minute.

Solution by William W. Johnson, Cleveland, Ohio.

The unit stress in the flywheel rim due to centrifugal force is found as follows: In the general equation of force, $F = Ma$, the acceleration a in the present instance is v^2/r, in which v = velocity of rim in feet per second, R = radius of rim in feet, W = weight of rim in pounds, and g = acceleration due to gravity. Hence, centrifugal force = $F = Mv^2/R = Wv^2/gR$.

The resultant of half of this force tends to disrupt one half of the rim from the other half. This rupture is resisted by the two sections of the rim at each end of the diameter.

The total tension T in a cross-section of the rim is to one half the sum of half of the radial forces as the diameter of the flywheel is to half its circumference. Therefore,

$$T = F/2\pi = Wv^2/2\pi Rg.$$

Let $W = 2\pi RAw/144$, in which A = area of cross-section of rim in square inches, and w = weight of rim material in pounds per cubic foot.

Then, $T = Awv^2/144g$. Let $T = AS$, where S = stress per unit area of cross-section of rim. Then, $S = wv^2/144g$.[1]

Put $v = \pi Dn/60$, where D = mean diameter of rim in feet, and n = number of revolutions per minute, we have

$$S = \frac{w(\pi Dn)^2}{60^2 \times 144g}. \qquad (A)$$

Solving for n, we have

$$n = \frac{720}{\pi D}\sqrt{\frac{Sg}{w}}. \qquad (B)$$

Putting, $D = 21$, $n = 100$, $w = 448$, $S = 6{,}000$, and $g = 32.16$ in formulas (A) and (B) we find

Answer (a): $S = 1{,}170$ lbs. per sq. in., and

Answer (b): $n = 226$ revolutions per min.

Also solved by J. B. Reynolds.

334. Proposed by HORACE OLSON, Chicago, Illinois.

A particle of elasticity e is projected with velocity v at an angle φ with a plane inclined to the horizontal at an angle ψ; its plane of motion is perpendicular to the inclined plane. Show that after $2v \sin \varphi/g(1 - e) \cos \psi$ seconds it will cease to rebound and will move along the plane with an initial velocity $v \cos \varphi - 2v \sin \varphi \tan \psi/1 - e$ and a uniform acceleration $g \sin \psi$ down the plane. Cf. Routh, Dyn. of a part., 1898, p. 90-1.

Solution by Jos. B. Reynolds, Lehigh University.

The acceleration of the particle may be resolved into a component of $g \cos \psi$ perpendicular to the plane and a component $g \sin \psi$ down the plane. Let v_1 be the initial component of the velocity of the particle perpendicular to the plane, v_2 the component of velocity perpendicular to the plane after the first rebound, v_3 after the second rebound, etc. Also, let t_1 be the time from the instant of projection until the first impact, t_2 the time from the first impact until the second, t_3, from the second until the third, etc. Then

$$v_2 = ev_1, \qquad v_3 = ev_2 = e^2v_1, \quad \text{etc.}$$

$$t_1 = \frac{2v_1}{g \cos \psi}, \qquad t_2 = \frac{2v_2}{g \cos \psi}, \qquad t_3 = \frac{2v_3}{g \cos \psi}, \quad \text{etc.}$$

So that if t is the time before the ball ceases to rebound,

$$t = t_1 + t_2 + t_3 + \cdots = \frac{2v_1}{g \cos \psi}(1 + e + e^2 + \cdots + e^n)_{n=\infty} = \frac{2v_1}{(1 - e)g \cos \psi}$$

$$= \frac{2v \sin \varphi}{(1 - e)g \cos \varphi}, \qquad \text{since} \qquad v_1 = v \sin \varphi.$$

[1] S is independent of the radius and depends only on the rim velocity.

Since the velocity along the plane is not affected by the impacts the velocity v' at the end of time, t, will be upward along the plane end of magnitude

$$v' = v\cos\varphi - g\sin\psi t$$
$$= v\cos\varphi - \frac{2v\sin\varphi\tan\psi}{(1-e)}, \qquad \text{since} \qquad t = \frac{2v\sin\varphi}{(1-e)g\cos\psi}.$$

Also solved by the PROPOSER.

335. Proposed by HAROLD T. DAVIS, Colorado Springs, Colorado.

A heavy particle is projected upwards with a velocity V in a medium resisting as the nth power of the velocity. Prove that the elevation of the particle when the velocity downwards is V is equal to LT, where L is the limiting velocity and T is the time in which the particle falling from rest in the medium will acquire a velocity V^2/L.

SOLUTION BY JOS. B. REYNOLDS, Lehigh University.

As the solution will show, this problem should read as follows: A heavy particle is projected upwards with a velocity V in a medium resisting as the nth power of the velocity. Prove that the whole space (up and down) described when the downward velocity is V is equal to LT, where L is the limiting velocity and T is the time in which the particle falling from rest in the medium will acquire a velocity V^2/L. M. J. M. 58-60.

The equation of motion for the particle upward is

$$-v\frac{dv}{ds} = g + \mu v^n \tag{1}$$

μ being the proportionality factor, and the equation of motion for the particle downward is

$$v\frac{dv}{ds} = g - \mu v^n = \frac{dv}{dt}. \tag{2}$$

When the particle has acquired the limiting velocity L its acceleration is zero. So by equation (2) we have $g/\mu = L^n$. Also by (2), when falling in the medium,

$$dt = \frac{dv}{g-\mu v^n}, \qquad \text{and} \qquad \therefore\ T = \int_0^{V^2/L}\frac{dv}{g-\mu v^n} = \frac{1}{\mu}\int_0^{V^2/L}\frac{dv}{L^n - v^n};$$

whence,

$$LT = \frac{L^{n+1}}{g}\int_0^{V^2/L}\frac{dv}{L^n - v^n}. \tag{3}$$

From (1), in the upward motion,

$$ds = -\frac{vdv}{g+\mu v^n};$$

whence, if the particle rises to a height h before coming to rest,

$$h = -\int_V^0\frac{vdv}{g+\mu v^n} = \int_0^V\frac{vdv}{g+\mu v^n}.$$

When falling, by (2),

$$ds = \int\frac{vdv}{g-\mu v^n},$$

so that if it has fallen a distance h_1 when the velocity is V, we have

$$h_1 = \int_0^V\frac{vdv}{g-\mu v^n}.$$

Then the total distance described (up and down) is

$$h + h_1 = \int_0^V\left(\frac{1}{g-\mu v^n} + \frac{1}{g+\mu v^n}\right)vdv = \int_0^V\frac{2gvdv}{g^2 - \mu^2 v^{2n}},$$

or

$$h + h_1 = \frac{g}{\mu^2}\int_0^V\frac{2vdv}{L^{2n} - V^{2n}} = \frac{g}{\mu^2 L^n}\int_0^V\frac{2vdv}{L^n - (V^2/L)^n}.$$

In this integral, if we let $z = v^2/L$, we shall have

$$h + h_1 = \frac{L^{n+1}}{g}\int_0^{V^2/L} \frac{dz}{L^n - z^n},$$

but this is identical with the integral of (3). Hence $h + h_1 = LT$.

Note.—The incorrectness of the statement of this problem was also pointed out by Professor H. S. Uhler for the special case $n = 2$. However, if in his solution we take the value he finds for h, the height to which the particle rises and the distance z' through which it falls from rest, we have

$$h + z' = \frac{1}{2k}\log\left(1 + \frac{k}{g}V^2\right) + \frac{1}{2k}\log\left(\frac{L^2}{L^2 - V^2}\right) = \frac{1}{2k}\log\left(\frac{L^2 + V^2}{L^2 - V^2}\right),$$

where $\sqrt{g}/\sqrt{k} = V$, the limiting velocity. But he showed that

$$LT = \frac{1}{2k}\log\left(\frac{L^2 + V^2}{L^2 - V^2}\right).$$

Hence, $h + z' = LT$, a result agreeing with that of Professor Reynolds. EDITORS.

NUMBER THEORY.

250. Proposed by JOSEPH E. ROWE, State College, Pa.

Show by comparatively elementary means that the equation $x^{2n} + y^{2n} = z^{2n}$ is impossible of solution in positive integers x, y, z and n, unless at least one of the integers x, y, $z \equiv 0 \pmod 3$. In particular, consider the case $n = 1$.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

Since any number must be of one of the forms $3k$, $3k + 1$, $3k - 1$, its square must be of the form $3l$ or $3l + 1$. Consequently, any perfect square, $x_1^{2n} \equiv 0$ or $1 \pmod 3$; it is clear that the three quantities in the given equation can not all be congruent to 1 (mod 3); in fact it is evident that either x^{2n} or y^{2n} must be congruent to 0 (mod 3), as otherwise the left-hand member of the equation would be divisible by 3 with remainder 2, and the right-hand member by 3 with remainder 1.

The case $n = 1$ may be handled directly by means of the known solution of the equation $x^2 + y^2 = z^2$, namely, $x = 2mn$, $y = m^2 - n^2$, $z = m^2 + n^2$ (we are supposing that x, y and z are prime to each other) where one of the two quantities n, m is even and the other odd. If either m or n is divisible by 3, x is also; if neither, then as we saw above $m^2 \equiv 1 \pmod 3$, $n^2 \equiv 1 \pmod 3$, and, consequently, $y = m^2 - n^2 \equiv 0 \pmod 3$.

Also solved by C. C. YEN, H. C. FEEMSTER, H. N. CARLETON, and J. W. CLAWSON.

251. Proposed by HERMAN ROLAND KATNICK, Chicago, Ill.

Determine the character of the positive integer n so that the Diophantine system

$$z + n = x^2, \qquad z - n = y^2$$

shall have an integral solution; and exhibit a method for finding all the values of x, y, z for a given n of such character.

SOLUTION BY THE PROPOSER.

[Mr. Katnick died suddenly shortly after this problem was offered for publication. The solution given below is a modified and abridged form of one offered by the proposer at the time the problem was submitted. EDITORS.]

From the given equations we have $x^2 - y^2 = 2n$ or $(x + y)(x - y) = 2n$. If we denote $x + y$ by a and $x - y$ by b we have

$$(1) \qquad x + y = a, \qquad x - y = b, \qquad 2n = ab.$$

Hence it is necessary that $2x = a + b$, $2y = a - b$, while at least one of the numbers a, b is even (since $ab = 2n$). Hence, both a and b are even since x and y are integers. Then we may put

$a = 2\alpha$, $b = 2\beta$, where α and β are integers. Then from (1) we have

$$x = \alpha + \beta, \qquad y = \alpha - \beta, \qquad n = 2\alpha\beta. \tag{2}$$

If these values of x, y, n are substituted in the original equations we find that

$$z = \alpha^2 + \beta^2. \tag{3}$$

From these computations one readily concludes as follows:

A necessary and sufficient condition on the positive integer n is that it shall be even. All non-negative solutions for a particular value of the even integer n may be found as follows: Separate $n/2$ in every way possible into the product of two positive integral factors α and β such that $\alpha \geqq \beta$. For each such separation the corresponding values of x, y, z are given by equations (2) and (3).

It is obvious that the solution is unique when and only when $n/2$ is a prime number.

Also solved by C. F. GUMMER, ELIJAH SWIFT, HORACE OLSON, E. B. ESCOTT, J. L. RILEY, C. E. GITHENS and V. M. SPUNAR.

252. Proposed by E. J. MOULTON, Northwestern University.

(*A*) Show that the number of integers x on the interval $10^r \leqq x < 10^{r+1}$ which do not contain the digit 1 at least p times, $p \leqq r$, is

$$9 \cdot \{[\text{first } p \text{ terms of expansion of } (9+1)^r] - {}_rC_{p-1} \cdot 9^{r-p}\},$$

where ${}_rC_{p-1}$ is the coefficient of x^{p-1} in the expansion of $(1+x)^r$.

(*B*) Show that the number of integers x on the interval $10^r \leqq x < 10^{r+1}$ which do not contain the digit 0 at least p times, $p \leqq r$, is

$$9 \cdot [\text{first } p \text{ terms of expansion of } (9+1)^r].$$

SOLUTION BY C. C. YEN, Tangshan, North China.

The proof of (*B*), being simpler, is given first.

(*B*) The total number of integers x on the interval $10^r \leqq x < 10^{r+1}$ is $(10^{r+1} - 10^r) = 9 \cdot 10^r$. If the whole interval be divided into nine subintervals, $n \cdot 10^r \leqq x < (n+1) \cdot 10^r$, $n = 1, 2, \cdots$ up to 9; then each subinterval will contain 10^r or $(9+1)^r$ integers.

Consider each of the subintervals. First, there is one integer which contains the digit 0 r times. Next there are ${}_rC_{r-1} \cdot 9$ integers which contain the digit 0 $(r-1)$ times; for out of the r digit places $(r-1)$ of them may be filled by 0's in ${}_rC_{r-1}$ ways; and the remaining place may be filled by any one of the digits 1, 2, 3, $\cdots$, 9, so that each of the ${}_rC_{r-1}$ ways gives rise to 9 such integers. In the same way, $(r-2)$ of the r places may be filled by 0's in ${}_rC_{r-2}$ different ways; and each of the two remaining places may be filled by any one of the digits 1, 2, $\cdots$ 9, so that each of the ${}_rC_{r-2}$ ways gives rise to 9^2 integers; hence, there are ${}_rC_{r-2} \cdot 9^2$ integers containing the digit 0 $(r-2)$ times, and so on. In general, p out of the r places may be filled by 0's in ${}_rC_p$ different ways, each of which gives rise to 9^{r-p} integers; the number of integers containing the digit 0 p times is, therefore, ${}_rC_p \cdot 9^{r-p}$. Hence, we have the result that in each subinterval there are

$${}_rC_p \cdot 9^{r-p} + \cdots + {}_rC_{r-2} \cdot 9^2 + {}_rC_{r-1} \cdot 9 + 1 \tag{1}$$

integers containing the digit 0 at least p times, $p \leqq r$. But expression (1) consists of the last $r - p + 1$ terms of the expansion of $(9+1)^r$; and, as we have seen, the number of integers in each subinterval is $(9+1)^r$. It follows therefore that the number of integers which do not contain the digit 0 at least p times $(p \leqq r)$ in each subinterval is given by the first p terms of the expansion $(9+1)^r$, and hence in the whole interval $10^r \leqq x < 10^{r+1}$, the number is

$$9 \cdot [\text{first } p \text{ terms of expansion of } (9+1)^r],$$

as was to be proved.

(*A*) To prove (*A*), let the interval $10^r \leqq x < 10^{r+1}$ be divided into subintervals as before. Then in each of the subintervals, except the first, similar reasonings show that the number of integers containing the digit 1 at least p times, $p \leqq r$, is expressed by (1). In the first subinterval, $10^r \leqq x < 2 \cdot 10^r$, the first digit of every integer x is 1, so that (1) gives the number of integers containing the digit 1 at least $(p+1)$ times. Now, out of the r digit places (beginning from the

second), $(p-1)$ may be filled by 1's in ${}_rC_{p-1}$ ways, and each of these different ways gives rise to $9^{r-(p-1)}$ integers containing the digit 1 p times. Hence, in the first subinterval the number of integers containing 1 at least p times is equal to the sum of expression (1) and ${}_rC_{p-1}\cdot 9^{r-p+1}$. And, therefore, the number of integers in the interval $10^r \leqq x < 10^{r+1}$ which do not contain the digit 1 at least p times, $p \leqq r$, is

$$9\cdot[\text{first } p \text{ terms of expansion of } (9+1)^r] - {}_rC_{p-1}\cdot 9^{r-p+1},$$

or

$$9\cdot\{[\text{first } p \text{ terms of expansion of } (9+1)^r] - {}_rC_{p-1}\cdot 9^{r-p}\}.$$

253. Proposed by HERBERT N. CARLETON, West Newbury, Massachusetts.

Prove that $n^{2k+3} - n^{2k} \equiv 0 \pmod{20}$ for integral values of n and k.

SOLUTION BY R. M. MATHEWS, Riverside, California.

$$n^{2k+3} - n^{2k} = n^{2k}(n^2-1)(n^2+1)(n^4+1)$$

When n is even, $n^{2k} \equiv 0 \pmod 4$. When n is odd, $n^2 - 1 \equiv 0 \pmod 4$.

Next, n being an integer must be of the form $5m$, $5m \pm 1$, or $5m \pm 2$.

For n of the form $5m$, $n^{2k} \equiv 0 \pmod 5$; for n of the form $5m \pm 1$, $n^2 - 1 \equiv 0 \pmod 5$; and for n of the form $5m \pm 2$, $n^2 + 1 \equiv 0 \pmod 5$.

Hence, $n^{2k+3} - n^{2k} \equiv 0 \pmod{20}$, n and k being integers. This is also true of $n^{2k+4} - n^{2k}$.

Also solved by O. S. ADAMS, W. J. THOME, ELIJAH SWIFT, E. B. ESCOTT, C. C. YEN, and the PROPOSER.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence.

REPLIES.

34. Given the mixed integral and functional equation

$$\int_{x=0}^{x=x} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{x}{2}\right) + f(x)\right],$$

to determine the function $f(x)$. This equation is of rather fundamental practical value as it has to do with the most general solid whose volume is given by the prismatoid formula.

REMARK BY S. A. COREY, Albia, Iowa.

The prismoidal formula gives the exact value of this integral whenever the fourth derivative of $f(x) = 0$. This was shown in an article entitled "Certain Integration Formulæ Useful in Numerical Computation" in Vol. XIX, Nos. 6 and 7, of this MONTHLY, in which formula (1r) is the prismoidal formula including an expression for the remainder term.

$f(x) = Ax^3 + Bx^2 + Cx + D$ is, therefore, the most general value of the function $f(x)$ for which the prismoidal formula gives the exact value for all values of x.

28. Is it possible to obtain $\int \cos\theta^2 d\theta$ without expanding $\cos\theta^2$? If it is not, can some interesting properties of this integral be determined by treating it as a special function?

REPLY BY OSCAR SCHMIEDEL, Bellevue College, Bellevue, Nebraska.

The four reduction formulæ

$$\int \theta^m \sin\theta^n\, d\theta = \frac{1}{m+1}\theta^{m+1}\sin\theta^n - \frac{n}{m+1}\int \theta^{m+n}\cos\theta^n\, d\theta,$$

$$\int \theta^m \sin\theta^n\, d\theta = -\frac{1}{n}\theta^{m+1-n}\cos\theta^n + \frac{m+1-n}{n}\int \theta^{m-n}\cos\theta^n\, d\theta,$$

$$\int \theta^n \cos\theta^n\, d\theta = \frac{1}{m+1}\theta^{m+1}\cos\theta^n + \frac{n}{m+1}\int \theta^{m+n}\sin\theta^n\, d\theta,$$

$$\int \theta^n \cos\theta^n\, d\theta = \frac{1}{n}\theta^{m+1-n}\sin\theta^n - \frac{m+1-n}{n}\int \theta^{m-n}\sin\theta^n\, d\theta,$$

give rise to the following four, which, like those above, give two developments of $\int \theta^m \sin\theta^n d\theta$ and $\int \theta^m \cos\theta^n\, d\theta$ each:

$$\int \theta^m \sin\theta^n\, d\theta = -\frac{\theta^{m+1}}{m+1}\sum_{t=0}^{t=r} n^t \prod_{s=1}^{s=t}\frac{1}{m+sn+1}\cdot\theta^{tn}\sin\left(t\frac{\pi}{2}-\theta^n\right)$$
$$- n^{r+1}\prod_{s=0}^{s=r}\frac{1}{m+sn+1}\cdot\int\theta^{m+(r+1)n}\cos\left(r\frac{\pi}{2}-\theta^n\right)d\theta,$$

$$\int \theta^m \sin\theta^n\, d\theta = \frac{\theta^{m+1}}{m+1}\sum_{t=1-r}^{t=-1} n^t \prod_{s=1+t}^{s=0}(m+sn+1)\cdot\theta^{tn}\sin\left(t\frac{\pi}{2}-\theta^n\right)$$
$$- n^{s-r}\prod_{s=1-r}^{s=-1}(m+sn+1)\cdot\int\theta^{m+(1-r)n}\cos\left(r\frac{\pi}{2}+\theta^n\right)d\theta,$$

$$\int \theta^m \cos\theta^n\, d\theta = \frac{\theta^{m+1}}{m+1}\sum_{t=0}^{t=r} n^t \prod_{s=1}^{s=t}\frac{1}{m+sn+1}\cdot\theta^{tn}\cos\left(t\frac{\pi}{2}-\theta^n\right)$$
$$- n^{1+r}\prod_{s=0}^{s=r}\frac{1}{m+sn+1}\cdot\int\theta^{m+(1+r)n}\sin\left(r\frac{\pi}{2}-\theta^n\right)d\theta,$$

$$\int \theta^m \cos\theta^n\, d\theta = -\frac{\theta^{m+1}}{m+1}\sum_{t=1-r}^{t=-1} n^t \prod_{s=1+t}^{s=0}(m+sn+1)\cdot\theta^{tn}\cos\left(t\frac{\pi}{2}-\theta^n\right)$$
$$+ n^{1-r}\prod_{s=1-r}^{s=-1}(m+sn+1)\cdot\int\theta^{m+(1-r)n}\sin\left(r\frac{\pi}{2}+\theta^n\right)d\theta,$$

where r is everywhere a positive integer.

If in any of these, one integral may be made to vanish, by taking r large enough, then the indicated sum will be the development of the other integral. Thus, making $m = 0$, $n = 2$, in the third of the set, we obtain for the integral of $\cos\theta^2$ the convergent series

$$\int \cos\theta^2\, d\theta = \theta \sum_{t=0} 2^t \prod_{s=1}^{s=t} \frac{1}{2s+1} \theta^{2t} \cos\left(t\frac{\pi}{2} - \theta^2\right)$$

$$= \cos\theta^2 \left(\theta - \frac{2^2}{1\cdot 3\cdot 5}\theta^5 + \frac{2^4}{1\cdot 3\cdot 5\cdot 7\cdot 9}\theta^9 - + \cdots\right)$$

$$+ \sin\theta^2 \left(\frac{2}{1\cdot 3}\theta^3 - \frac{2^3}{1\cdot 3\cdot 5\cdot 7}\theta^7 + - \cdots\right).$$

Two interesting special cases deserve mention.

When the exponent of the coefficient is less by unity than that of the argument of the function, the integral assumes a known form; and the value of the other integral becomes known. Hence, expressions of the type $\theta^{rn-1} \sin\theta^n\, d\theta$ are directly integrable in finite form by these formulæ. If in addition n takes the special form $1/r$, the integral of $\sin\theta^{1/r} d\theta$ is found, that is, the integral of sine or cosine of any root of a variable is found in finite form.

Thus, taking the first of the four formulæ, making $m = n - 1$, and transforming a little; and after this, making $n = 1/r$, we obtain the formulæ:

$$\int \theta^{rn-1} \cos\left(r\frac{\pi}{2} - \theta^n\right) d\theta = -\, n^{-1} \prod_{s=r-1}^{s=1} s \cdot \sum_{t=1}^{t=r} \prod_{s=1}^{s=t-1} s^{-1} \cdot \theta^{(t-1)n} \sin\left(t\frac{\pi}{2} - \theta^n\right),$$

$$\int \cos\left(r\frac{\pi}{2} - \theta^{1/r}\right) d\theta = -\, r \prod_{s=1}^{s=r-1} s \cdot \sum_{t=1}^{t=r} \prod_{s=1}^{s=t-1} s^{-1} \cdot \theta^{(t-1)1/r} \sin\left(t\frac{\pi}{2} - \theta^{1/r}\right).$$

As examples, in these formulas making $n = 2$, $r = 4$, we have

$$\therefore \int \theta^7 \cos\theta^2\, d\theta = -\, 3\left[\cos\theta^2\left(1 - \frac{\theta^4}{2}\right) + \sin\theta^2\left(\theta^2 - \frac{\theta^6}{3!}\right)\right],$$

$$\int \cos\theta^{1/4} = -\, 4!\left[\cos\theta^{1/4}\left(1 - \frac{\theta^{1/2}}{2}\right) + \sin\theta^{1/4}\left(\theta^{1/4} - \frac{\theta^{3/4}}{3!}\right)\right].$$

DISCUSSIONS.

I. Relating to a Curve with Unusual Properties.

By Jos. B. Reynolds, Lehigh University.

The curve

$$y = \frac{a^2 x}{x^2 + a^2}$$

presents some unusual properties. Considering the part of the curve to the right of the y-axis (see the figure) we have for the area between it and the x-axis

$$A = \int_0^\infty y dx = a^2 \int_0^\infty \frac{x dx}{x^2 + a^2} = \frac{a^2}{2} \log(x^2 + a^2) \Big]_0^\infty = \infty.$$

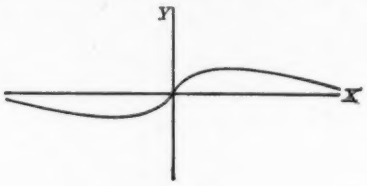
Yet the volume generated by revolving this area about the x-axis is

$$V = \pi \int_0^\infty y^2 dx = \pi a^4 \int_0^\infty \frac{x^2 dx}{(x^2 + a^2)^2} = \frac{\pi^2 a^3}{4};$$

that is, the revolution of an infinite area gives a finite volume. Again, the ordinate of the center of gravity of this area is

$$\bar{y} = \frac{\int_0^\infty dx \int_0^{a^2x/(x^2+a^2)} y dy}{\int_0^\infty dx \int_0^{a^2x/(x^2+a^2)} dy} = \frac{\frac{\pi a^3}{8}}{\infty} = 0;$$

that is, although the area lies entirely above the x-axis and is indefinitely large,



its center of gravity is indefinitely close to that axis.

By the theorem of Pappus we have

$$2\pi\bar{y}A = V,$$

whence we have an indeterminate form evaluated

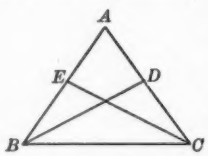
$$2\pi(0)\infty = \frac{\pi^2 a^3}{4}.$$

II. Relating to a Demonstration of a Geometrical Theorem.

By William E. Heal, Washington, D. C.

If the bisectors of two angles of a triangle are equal, the triangle is isosceles.

This is a very celebrated theorem and has been demonstrated in many ways. It was proposed in the Monthly as problem 42 and several demonstrations were published in Vol. II, pp. 157 and 189–191. Of these proofs all but one were by indirect methods; that is, by use of the *reductio ad absurdum*. The following direct method of proof was communicated to the writer some months ago by Dr. Artemas Martin of the U. S. Coast Survey and seems to be eminently worthy of preservation.



If $BD = CE$ we are to prove that $AB = AC$.

We have by a well-known theorem

$$AB \times BC = (AE + BE)BC = BD^2 + AD \times DC, \tag{1}$$

$$AC \times BC = (AD + DC)BC = CE^2 + AE \times EB. \tag{2}$$

If $BD = CE$ these become

$$(AE + BE)BC = BD^2 + AD \times DC, \tag{3}$$

$$(AD + DC)BC = BD^2 + AE \times EB. \tag{4}$$

Subtracting (3) from (4) and transposing,

$$AE \times BC + BE \times BC + AE \times BE = AD \times DC + AD \times BC + DC \times BC. \quad (5)$$

We also have

$$AB : BC = AD : DC,$$

$$AC : BC = AE : EB,$$

$$\therefore \quad BC \times AD = AB \times DC = (AE + BE)DC, \quad (6)$$

$$BC \times AE = AC \times EB = (AD + DC)EB. \quad (7)$$

Substituting these values of $BC \times AD$ and $BC \times AE$ in (5) and dropping $BE \times CD$ from both sides we have

$$AD \times EB + BC \times EB + AE \times EB = AD \times DC + AE \times DC + BC \times DC.$$

Or

$$(AD + BC + AE)EB = (AD + AE + BC)DC.$$

$$\therefore \quad EB = DC.$$

Substituting $EB = DC$ in (5) we get, after dropping $EB \times BC$ from both sides,

$$AE \times BC + AE \times BE = AD \times EB + AD \times BC$$

or

$$(BC + BE)AE = (BC + BE)AD.$$

Hence

$$AE = AD.$$

Therefore

$$AB = AC.$$

III. Concerning Huntington's *Continuum and Other Types of Serial Order.*[1]

By Lester S. Hill, Princeton University.

If A_1 is any subclass of an ordered class A, then all elements a of A which are ordinally less than every element of A_1 ($a < A_1$) constitute a lower segment of A; and all elements $a > A_1$ constitute an end segment. If A_1 and A_2 are two subclasses with $A_1 < A_2$ then all elements a for which $A_1 < a < A_2$ constitute a mid segment. Of course A_1 or A_2, or each of them, may reduce to a single element of A.

The order type of A is continuous if

(D) A is dense: between every two elements of A lie other elements;

(G) A is gap-free: If A' is a lower segment, A'' an end segment, and $A = A' + A''$, then A' has a last element or else A'' has a first. ($A = A' + A''$ implies $A' < A''$.)

The ordered class A of real numbers satisfies (D), (G) and also

(F_0) A is denumerably framed: A contains a *denumerable* subclass B which is dense in A; between every two elements of A lie elements of B.

[1] See Book Review on page 325 of this issue.

The more general continuous types for which Huntington has space to cite examples satisfy (D), (G) and the following weakened form of (F_0),—

(F) A is denumerably framed by segments: on every segment of A lie segments which are denumerably framed.

It may therefore be of interest to illustrate a simple type of continuous order satisfying neither (F_0) nor (F).

§ I. Let $H = \{h\}$ be the class of all numerical sequences of type ω

$$h = (e_1, e_2, e_3, \cdots),$$

where e_n is any real number for every finite index n, only finite ordinal numbers occurring as indices.

Then, $h' = (e_1', e_2', e_3', \cdots)$ and $h'' = (e_1'', e_2'', e_3'', \cdots)$

being two elements of H we define

$h' < h''$ (h' less than h'') if, from some index on, $e_n' < e_n''$,
$h' > h''$ (h' greater than h'') if, " " " " $e_n' > e_n''$,
$h' = h''$ (h' equal to h'') if, " " " " $e_n' = e_n''$.

If one of these three relations holds we call h' and h'' comparable; otherwise h and h' are incomparable.

Observe that if $h' \leqq h''$ and $h'' \leqq h'''$ then $h' \leqq h'''$.

§ II. We omit the proof that there exist subclasses $K = \{k\}$ of H, every k being an h, with the properties of closure:

(α) K contains no pair of incomparable, and no pair of equal sequences.

(β) The class $H - K$ contains no sequence which is comparable with every k but equal to no k.

§ III. $K = \{k\}$, which is, to to speak, a maximum (closed) subclass of serially ordered sequences in H, has the striking ordinal property:

If K' and K'' are denumerable subclasses of K such that $K' < K''$, then K contains an infinite subclass K''' such that $K' < K''' < K''$.

For, since K' and K'' are denumerable, we can arrange them

$$K' = (k_1'k_2'k_3'\cdots), \qquad K'' = (k_1''k_2''k_3''\cdots),$$

k_1' being the first sequence of K' in this arrangement, etc. We have, more in detail,

$$k_1' = (k_{11}', k_{12}', k_{13}', \cdots); \qquad k_1'' = (k_{11}'', k_{12}'', k_{13}'', \cdots)$$

$$k_2' = (k_{21}', k_{22}', k_{23}', \cdots); \qquad k_2'' = (k_{21}'', k_{22}'', k_{23}'', \cdots)$$

. . . . . .

and, of course, every k_n' is less than every k_m'' ($K' < K''$).

We can now determine a sequence of indices $n_1 < n_2 < n_3 < \cdots$ such that

$$k_{1n}' < k_{1n}'' \qquad (n \geqq n_1)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$k_{1n}', k_{2n}', \cdots, k_{mn}' < k_{1n}'', k_{2n}'', \cdots, k_{mn}'' \qquad (n \geqq n_m)$$

and, obviously, the sequence $h = (e_1, e_2, e_3, \cdots)$ where the e_n are any real numbers such that

$$k_{1n}' < e_n < k_{1n}'' \qquad (n_1 \leqq n < n_2)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$k_{1n}', k_{2n}', \cdots, k_{mn}' < e_n < k_{1n}'', k_{2n}'', \cdots, k_{mn}'' \quad (n_m \leqq n < n_{m+1})$$

is an element of H greater than K' and less than K''. Therefore, if we suppose the mid segment of K determined by $K' < K''$ to be zero (that is, without elements), it is clear that h is comparable with every element of K, and is itself an element of K; and we have a contradiction. In this argument we use the observation made at the end of § I.

If K' or K'', or each of these classes, reduces to a finite class or to a single element of K, the reader may supply the appropriate argument and obtain the same conclusion as above. Hence K is certainly dense, satisfying (D).

§ IV. Hausdorff defined K and proved that its potency is that of the real number continuum.

We need not trouble to ascertain whether K satisfies condition (G). For we can obtain from K very easily a class J ordered according to (D) and (G), and having the property discussed in § III.

Let $J = \{j\}$ be the class of all those lower segments in K which are without last element; and define $j_1 < j_2$ if the segment j_2 contains elements of K not occurring in j_1. A very hasty reading of Huntington's book will enable the student to see at once that J, thus ordered, satisfies (D) and (G); and that it contains a subclass L which is dense in J and similar (ordinally equivalent) to K: namely, the class of lower segments in K corresponding to partitions $K = K' + K''$ $(K' < K'')$, in which K' has no last element, but K'' has a first.

The density of L in J and the ordinal equivalence of L and K furnish an easy proof (based on that property of K discussed in § III) for the statement:

J contains no denumerably framed segment. While continuous it is widely divergent from the continuous types cited in the manual under review. It has a property meaning more than the mere absence of the property (F) which characterizes those types.

The reader may again follow indications by Hausdorff in the construction of number systems "Gröszensysteme" which are serially ordered, but non-Archimedean, and otherwise different from the familiar system of real numbers.

Let $H = \{h\}$ be, as before, the class of all real number sequences; and, if $h' = (e_1', e_2', e_3', \cdots)$, $h'' = (e_1'', e_2'', e_3'', \cdots)$ are two elements of H define:

$$h' \pm h'' = (e_1' \pm e_1'', \quad e_2' \pm e_2'', \cdots)$$

$$h' \gtrless h'' \text{ for } e_m' = e_m'' \ (m < n) \text{ and } e_n' \gtrless e_n''.$$

H is herewith furnished with serial order; and with an associative and commutative addition having the property: $h_1 + h_2 > h_1 + h_3$ if $h_2 > h_3$. The symbols mh, h/n, mh/n where m, n are positive or negative integers have an obvious meaning; and H contains a definite element described by $r_1h_1 + r_2h_2 + \cdots + r_nh_n$ where $h_1 \cdots h_n$ are any elements of H and $r_1r_2 \cdots r_n$ are any rational *real* numbers.

Like the Gröszensystem upon which Veronese bases his geometry of the intuitional continuum, H is not Archimedean; if $h_1 < h_2$ are an arbitrary pair of its elements, we can not assert that for some positive integer n, $nh_1 > h_2$; in fact such an assertion would be false in the obvious instance

$$h_1 = (0, 1, 0, 0, \cdots), \qquad h_2 = (1, 0, 0, 0, \cdots).$$

H contains relative "infinites" and "infinitesimals" of all finite orders; in striking contrast with the real number system, in which no constant number is infinitesimal or infinite with respect to any other constant.

Desiring a continuously ordered Gröszensystem we proceed again to the class $J = \{j\}$ of all lower segments in H without last element; and define $j_1 < j_2$ when the segment j_2 contains elements of H not occurring in j_1. As before, we obtain a continuously ordered J (but of order type very different from that of the former J) containing a subclass L which is dense in J and ordinally equivalent to H. That the extension of H to J is possible appears in the obvious fact that H is dense; that an extension is necessary is shown by the simple partition $H = H' + H''$ $(H' < H'')$ where H' contains all elements of H ordinally less than some $(0, e, 0, 0, \cdots)$, H'' all elements ordinally greater than some $(e, 0, 0, 0, \cdots)$ as e assumes all real values > 0: a partition which explodes condition (G).

The means of defining suitably an operation of addition in J is suggested by Dedekind's procedure in extending the class of rational numbers to the class of real numbers (cf. C. Jordan, *Cours d'Analyse*, Vol. I). Let $j_1 < j_2$ be elements of J representing the lower segments H_1 and H_2 of H. We define $j_1 + j_2$ as that lower segment (clearly without last element) in H which consists of all sums $h_1 + h_2$ of an element in H_1 and an element in H_2.

The properties of addition as defined in H validate this form of definition; and establish the desirable relation: if $l_1 < l_2$ of L correspond to $h_1 < h_2$ of H, then $l_1 + l_2$ is an element of L and corresponds to $h_1 + h_2$ of H. We have in L a class which is equivalent to H with respect to order and also with respect to addition. Since L is thus completely isomorphic with H, and is moreover dense in J, we are more than enabled to argue that J, like H, is non-Archimedean; indeed the Archimedean property involves only a relation between the ordering and the adding in a system with order and addition.

Hausdorff has given a general analysis of a large class of Gröszensysteme with order and addition which are non-Archimedean. He defined the particular case H explicitly but is not responsible for shortcomings which may be found in J.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Ind.

The permanent address of the Secretary, Professor W. D. CAIRNS, is now 27 King Street, Oberlin, Ohio.

At Hamilton College, Associate Professor W. M. CARRUTH has been promoted to a professorship of mathematics.

Professor L. E. DICKSON, of the University of Chicago, is lecturing at the University of California during the first semester.

Professor G. A. MILLER, of the University of Illinois, will contribute the mathematical article to the "American Year Book" for 1917.

Dr. W. H. BESANT, F.R.S., fellow of St. John's College and lecturer on mathematics, died on June 2, 1917, in his eighty-ninth year.

Professor H. HALPERIN, of Vanderbilt University, has been appointed assistant professor of mathematics at the University of Arkansas.

Professor C. A. BARNHART, of Carthage College, will be a member of the mathematical faculty of Colorado College during the coming year.

At the Pennsylvania State College Associate Professors J. H. TUDOR and H. F. STECKER have been promoted to professorships of mathematics.

Dr. W. V. N. GARRETSON, of the University of Michigan, has been appointed associate professor of mathematics at the Pennsylvania Military College.

The death of Dr. G. W. HARTWELL, for eight years professor of mathematics in Hamline University, St. Paul, Minn., occurred July 23 at Columbus, Montana. He was a charter member of the Association.

Dr. H. N. WRIGHT, formerly instructor in the University of California, has been appointed head of the department of mathematics at Whittier College, Whittier, Calif.

Mr. H. C. GOSSARD, assistant professor of mathematics at the University of Oklahoma, has been appointed a member of the mathematical staff in the Naval Academy at Annapolis.

Professor P. P. BOYD, of the University of Kentucky, has been appointed dean of the college of arts and sciences. Professor BOYD has also been made acting president of the university.

Professor G. H. SCOTT, for the fifteen years professor of mathematics and astronomy in Yankton College, Yankton, S. Dakota, has resigned to become principal of Benzonia Academy, Benzonia, Michigan.

Mrs. ELIZABETH BROWN DAVIS, a charter member of the Association, died at her home in Washington, D. C., in April, 1917. She was a computer in the Nautical Almanac division of the U. S. Naval Observatory.

At Washington University, St. Louis, Professor C. A. WALDO has retired from active service and Dr. W. H. ROEVER has been appointed acting head of the department of mathematics and promoted to a full professorship.

Mr. W. W. JOHNSON, 12013 Saywell Ave., Cleveland, Ohio, would like to learn from any reader of the MONTHLY where a copy of Steinhauser's *Berechnung zwanzigstelliger Logarithmen* (published by Carl Gerold's Sohns, Vienna) could be obtained.

The valuable paper by Professor A. R. CRATHORNE, of the University of Illinois, on "Required Mathematics," read before the Fourteenth Annual Conference of Kansas High Schools and Academies, March 16, 1917, appears in full in *School and Society*, July 7, 1917.

At recent meetings of the Academy of Sciences of Petrograd mathematical papers were presented by I. M. VINOGRADOV on "A new method of obtaining asymptotic expressions"; and by V. A. STEKLOV on "The approximation of functions by means of Cebysev's polynomials."

M. EMILE PICARD, professor of mathematics at the University of Paris, has been elected permanent secretary of the Paris Academy of Sciences, succeeding the late M. GASTON DARBOUX. Professor PICARD has also been appointed to represent the French government in the International Geodetic Association.

With a recent number of the *Revista de la Sociedad Matemática Española* the Spanish Mathematical Society has published a Supplement written by the president of the Society, Dr. ZOEL GARCIA DE GALDEANO, upon the development of mathematics under the title "Exposición sumaria de la matemática segŭn un nuevo método."

Mr. PHILIP E. B. JOURDAIN is the author of the reports in the quarterly journal, *Science Progress*, on "Recent Advances in Mathematics." In the July, 1917, issue he reports the joint action of the Association and the Society at the New York meeting in respect to possible assistance for the *Revue Semestrielle* and the *Fortschitte*.

When this issue reaches the readers the second summer meeting of the ASSOCIATION will have been ended. The program was practically as announced in the June MONTHLY except that some changes were finally made in respect to those who were to lead the various discussions. The full report of the meeting will appear in the October issue.

The *Rendiconti del Circolo Mathematico di Palermo* has announced that, on account of conditions due to the war, its publication will be suspended until the war is over. The editors of the MONTHLY wish to manifest their sympathy and to state that the exchange copy of the MONTHLY will be sent regularly as heretofore.

Science and Learning in France is the title of a volume of 454 pages recently issued by the Society for American Fellowships in French Universities. The mathematical editors are D. R. CURTISS, T. F. HOLGATE, E. H. MOORE, and E. B. WILSON. The Introduction appears under the two sub-headings "The Mind of France" and "The Intellectual Inspiration of Paris," written by CHARLES W. ELIOT and GEORGE E. HALE respectively, and a list of about one thousand sponsors follows the list of authors.

General WILLIAM H. BIXBY, U. S. Army, retired, a charter member of the Association, was on duty at Kansas City, Missouri, during June in charge of the Kansas City U. S. Engineer Office and its Missouri River improvements. At the end of June he was transferred to St. Louis as president of the Mississippi River commission in addition to his other work.

A communication from the Oxford University Press calls attention to an error in the price of Hill's "Development of the Arabic Numerals" as stated by the reviewer in the December, 1915, issue of the MONTHLY. It should be \$2.50 instead of \$1.00. Prices of books reviewed are quoted for the benefit of the MONTHLY readers, and it was supposed that this price was correctly given.

The sixth regular meeting of the Association of Mathematics Teachers of New Jersey was held at Rutgers college on May 26, under the presidency of Professor C. O. GUNTHER, of Stevens Institute of Technology. The program consisted of the "Final report of the committee on courses in trigonometry," by C. O. GUNTHER; "Preliminary report of the committee on high school courses," by A. W. BELCHER, of Newark high school; "Rectification of circular arcs," by SYDNEY SEIDLER, of Rutgers college; "Some applications of complex quantities," by C. O. GUNTHER; "Limits in elementary mathematics," by RICHARD MORRIS, of Rutgers College.

North Carolina has effected an organization of the teachers of secondary mathematics for the western portion of the state. The following officers have the work in hand: W. W. RANKIN, of the University of North Carolina, president; J. W. LASLEY, of the University of North Carolina, secretary-treasurer. This organization is the outgrowth of the feeling that the conditions in the secondary schools and colleges of the state would be bettered by a coöperation of the teachers. At the first meeting, held in Greensboro, April 13 and 14 the situation was gone over with care and plans were laid which have as their end better teaching of mathematics in North Carolina.

The special articles of expository or historical character in the September issue of the *Annals of Mathematics* are (1) "Minimal surfaces applicable to surfaces of revolution," by Dr. J. K. WHITTEMORE, of Yale University, and (2) "Ratio, proportion and measurement in the *Elements* of Euclid," by Professor HENRY B. FINE, of Princeton University. This is the second issue of the *Annals* under the new arrangement with the ASSOCIATION, the first issue being in June, 1917, which contained the expository article by Professor L. E. DICKSON on "Fermat's last theorem and the nature and theory of algebraic numbers."

The number of new subscribers to the *Annals* under the terms of coöperation with the ASSOCIATION is now (September 1) 350. The opportunity is still open at the half rate to members of the ASSOCIATION and to applicants for membership.

The great military preparations now engaging the United States have materially reduced the summer session attendance in many of the universities. The attendance during the coming year will, no doubt, be reduced in even a larger proportion. This will entail a reduction in the teaching force already partially discounted by the resignations of members of the faculties. A number of instructors and assistant professors of mathematics have enlisted or entered training camps. The following items, manifestly incomplete, have thus far been reported to the Editors: Mr. P. B. HILL, assistant in mathematics at Wabash College, and Mr. OWENS and Mr. FARRIS, assistants in mathematics at the Alabama Polytechnic Institute, were in the first Reserve Officers' Training Camps. Mr. J. F. CONNER, instructor in mathematics at the Catholic University of America, has been commissioned first lieutenant in the Bureau of Supplies and Accounts, United States Navy. Assistant Professor K. P. WILLIAMS, of Indiana University, has been commissioned captain of a company of field artillery recently mustered into the service of the nation. Professor PETER FIELD, of the University of Michigan, attended the first training camp at Fort Sheridan, Chicago, and is now a captain of ordnance in the coast artillery located at Fortress Monroe. Dr. WALTER L. HART, Benjamin Pierce Instructor at Harvard University, has been commissioned a second lieutenant in the field artillery section of the regular army. Mr. E. P. Hubble, candidate for the doctorate in astronomy and mathematics at the University of Chicago, has been commissioned captain of infantry, officers reserve corps, in the national army. Mr. LESTER S. HILL, instructor in mathematics at Princeton University, has been given a staff commission as ensign in the United States Navy and has been called into active service. Dr. R. W. BURGESS, of Brown University, is in the government service in the statistical department at Washington. Assistant Professor A. L. UNDERHILL, of the University of Minnesota, has a captain's commission in the Coast Artillery, and is now at Fort Constitution, Portsmouth, N. H.

AN APPEAL TO ALL MEMBERS OF THE ASSOCIATION:

Please report to Professor D. A. Rothrock, Chairman of the Committee on Notes and News, Indiana University, Bloomington, Indiana, all items within your knowledge, appropriate for the news column of the MONTHLY such as promotions, new appointments, club meetings, etc., and, particularly at this time, any facts concerning members of the ASSOCIATION (or others in the mathematical field) who have entered, or are about to enter, service for the government in any capacity. It is desired to make this column of genuine interest and value to the members, and to this end universal coöperation is needed. Please do this *at once* and from *time to time*.

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VOLUME XXIV OCTOBER, 1917 NUMBER 8

SECOND SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

The second summer meeting of the Association was held by invitation of Western Reserve University and Case School of Applied Science at Cleveland, Ohio, on Thursday and Friday, September 6–7, 1917, in conjunction with and following the summer meeting of the American Mathematical Society. There were 90 persons in attendance at the various sessions, including the following 72 members of the Association:

O. P. AKERS, Allegheny College.
FLORENCE E. ALLEN, University of Wisconsin.
R. B. ALLEN, Kenyon College.
L. D. AMES, University of Missouri.
FREDERICK ANDEREGG, Oberlin College.
R. C. ARCHIBALD, Brown University.
G. N. ARMSTRONG, Ohio Wesleyan University.

GRACE M. BAREIS, Ohio State University.
I. A. BARRETT, Chicago, Ill.
MRS. W. E. BECKWITH, College for Women, Western Reserve University.
H. F. BLICHFELDT, Stanford University.
J. W. BRADSHAW, University of Michigan.
R. W. BURGESS, Brown University.

W. D. CAIRNS, Oberlin College.
FLORIAN CAJORI, Colorado College.
W. M. CARRUTH, Hamilton College.
G. E. CARSCALLEN, Hiram College.
E. H. CLARKE, Hiram College.
G. R. CLEMENTS, U. S. Naval Academy.
BYRON COSBY, Kirksville (Mo.) State Normal School.
A. R. CRATHORNE, University of Illinois.
C. H. CURRIER, Brown University.

F. F. DECKER, Syracuse University.
R. M. DEMING, Case School of Applied Science.
L. W. DOWLING, University of Wisconsin.

JOHN EIESLAND, West Virginia University.
L. P. EISENHART, Princeton University.

T. M. FOCKE, Case School of Applied Science.
TOMLINSON FORT, University of Alabama.

M. G. GABA, Cornell University.
D. C. GILLESPIE, Cornell University.
O. E. GLENN, University of Pennsylvania.
C. F. GUMMER, Queen's University.

A. M. HARDING, University of Arkansas.
H. E. HAWKES, Columbia University.
E. R. HEDRICK, University of Missouri.
T. H. HILDEBRANDT, University of Michigan.
WILLIAM HOOVER, Ohio University (Retired).
E. V. HUNTINGTON, Harvard University.
W. A. HURWITZ, Cornell University.

R. A. JOHNSON, Western Reserve University.

O. D. KELLOGG, University of Missouri.
A. M. KENYON, Purdue University.